FIRST STEPS IN THE CALCULUS

GODFREY AND SIDDONS

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FIRST STEPS IN THE CALCULUS

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(BEING CHAPTERS XXIII TO XXVI OF *ELEMENTARY ALGEBRA* BY THE SAME AUTHORS, TOGETHER WITH A SET OF MISCELLANEOUS EXERCISES)

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PREFACE

THE fundamental ideas of the Calculus are now considered to form an essential part of a mathematical course for boys who are taking mathematics as part of their general education; these ideas are no longer reserved for specialists in mathematics.

The simplest elements of the Calculus now enter into various school examinations. The regulations for the Cambridge Local Examinations (1914) specify under Junior Algebra "The gradient of a graph." Under Senior Algebra it is stated that "for candidates for honours, easy questions will also be set on the more advanced parts of Algebra, together with elementary questions on problems involving differentiation and integration. The expressions to be differentiated and integrated will be limited to those whose terms are simple powers of the variables (avoiding the integration of $1/x$)." The regulations for the Oxford Local Examinations (1915) specify "the gradient of a graph" for Junior, Higher and Senior Elementary Algebra. The syllabus for Additional Mathematics for the Higher and School Certificates of the Oxford and Cambridge Joint Board specifies, under Algebra, "rate of change of a function and gradient of a graph." These extracts shew that Calculus has found its way into the non-specialist course. There are

many excellent elementary books on Calculus in existence; but, as far as we are aware, there are none that contain only the amount of matter that an average non-specialist can reasonably be expected to cope with; namely the ideas of gradient and rate of change as related to the differentiation and integration of simple powers of a variable. This is what the present modest volume contains: it may be useful to students who do not use the *Elementary Algebra* of which it forms a part.

There are several questions to be answered before the proper position of these topics in the education of the non-specialist can be ascertained and standardised. What amount of the infinitesimal calculus can reasonably be included in the non-specialist syllabus, and at what point will the mathematical specialist branch off? Can the average pupil grasp the meaning of the word "limit," or is it enough if he avoids gross error in this direction? To what extent are the initial difficulties of the calculus due to its special notation? If this is so, should an easier notation be used at first, and the standard notation introduced later? Or need the standard notation be introduced at all?

It is clear that the first chapter of the ordinary text-book on calculus does not provide a proper introduction to the subject for the average pupil; a much more gradual and leisurely approach is needed. The authors have worked on the assumption that the main ideas and applications of the calculus can be set forth without using functions more elaborate than x^3, x^2, x, $\frac{1}{x}$. The first two chapters in this book (Chaps. XXIII, XXIV) do not in fact go beyond rational integral functions of the second degree, and even within this limitation it is possible to give many practical

illustrations of maxima and minima, small increases, velocity and acceleration. In these two chapters the difficulties connected with the use of the standard notation are avoided. This notation however is used in Chapter XXV, as it will probably be useful to many students who need a minimum of calculus in connection with their subsequent study of other subjects. In Chapter XXVI the more obvious applications of integration are explained, the process being viewed as the solution of a differential equation; the integration sign is not used.

The thanks of the authors are due to the Controller of H.M. Stationery Office, the Director of Naval Education, and the O. and C. Joint Board for permission to include questions from various public examinations.

C. G.

A. W. S.

January 1914.

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CHAPTER XXIII

GRADIENT OF A GRAPH. DERIVED FUNCTIONS. [LINEAR AND QUADRATIC FUNCTIONS]

220. Gradient of a straight line. The word "gradient" is often used rather loosely, and for mathematical purposes we must give a precise meaning to this term. The slope or gradient of a road or railway can be stated in either of two ways. We may state the angle at which the road is inclined to the horizontal; this angle we will call the **angle of slope.** Or again, we may say that the **gradient** is, say, 1 in 100; which means that the road rises 1 foot in a length of 100 feet, or 1 yard in 100 yards, and so forth. Here again an ambiguity may occur, for the 100 feet may be measured either along the road, or horizontally. For mathematical purposes, we will decide to measure horizontally.

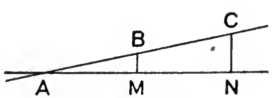

Fig. 32.

Thus, the gradient of ABC in fig. 32 is measured by the ratio BM : AM $= 1 : 5$; we might equally well take

$$\mathrm{CN} : \mathrm{AN} = 2 : 10 = 1 : 5.$$

221. In the case of a graph, the x axis takes the place of the horizontal. The gradient of a straight line graph is defined as the ratio

$$\frac{\text{increment of } y}{\text{increment of } x}.$$

It will be understood that the increment (i.e. increase or decrease) of y is that which *corresponds to* the increment of x.

It is necessary to distinguish between the cases of a line that goes *up* to the right, and a line that goes *down* to the right; and indeed the above definition does so distinguish. For in the former case (fig. 33 i) an increase of x gives rise to a corresponding

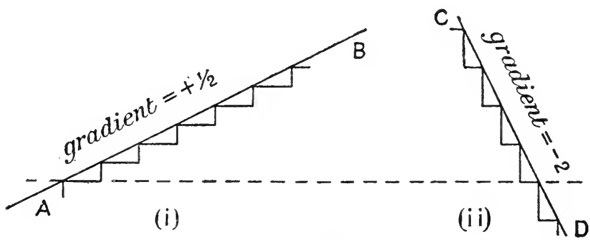

Fig. 33.

increase of y; in other words, a positive increment of x gives rise to a positive increment of y. Similarly a negative increment of x gives rise to a negative increment of y. Thus the ratio (increment of y : increment of x) is positive, and accordingly the gradient is positive.

In the latter case (fig. 33 ii) a positive increment of x gives rise to a negative increment of y; while a negative increment of x gives rise to a positive increment of y. In this case the ratio of the increments is negative, and the gradient is negative.

Thus in fig. 33 the gradient of AB is $+ 1/2$, that of CD is $-2/1$ or -2.

EXERCISE XXIII. a.

Find the gradients of the following straight lines (not more than 3 points need be plotted for each graph):—

1. (i) $y=2x$; (ii) $y=2x+1$; (iii) $y=2x+2$; (iv) $y=2x-1$; (v) $y=2x+c$ (c being a constant number).

2. (i) $y=\frac{1}{2}x$; (ii) $y=\frac{1}{2}x+1$; (iii) $y=\frac{1}{2}x+3$; (iv) $y=\frac{1}{2}x-2$; (v) $y=\frac{1}{2}x+c$.

3. (i) $y=-\frac{1}{3}x$; (ii) $y=-\frac{1}{3}x+2$; (iii) $y=-\frac{1}{3}x+\frac{1}{3}$; (iv) $y=-\frac{1}{3}x-2$; (v) $y=-\frac{1}{3}x+c$.

4. What can be stated about the gradient of all the different lines obtained by giving different values to c in $y=5x+c$?

222. To find the gradient of $y=mx+c$.

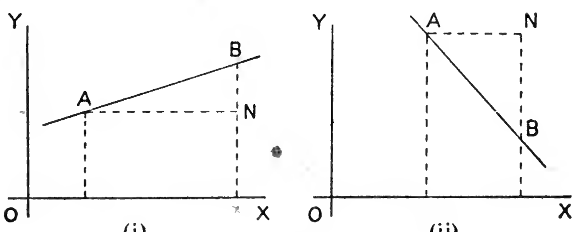

Fig. 34.

Let AB be the graph of $y=mx+c$; let the coordinates of A be (x_A, y_A); of B (x_B, y_B).

Then $y_A = mx_A + c$ ....(i),

$y_B = mx_B + c$ ....(ii).

The gradient is

$$\frac{NB}{AN} \text{ (having regard to sense)} = \frac{y_B - y_A}{x_B - x_A}.$$

But from equations (i) and (ii)

$$y_B - y_A = m(x_B - x_A).$$

$$\therefore \frac{y_B - y_A}{x_B - x_A} = m.$$

The gradient is therefore m.

Note that if we keep the same value for m and take different values for c, we have a series of parallel straight lines, all of gradient m.

In finding the gradient, the constant c makes no difference.

¶**Ex. 1.** What is the graph $y = c$? What is its gradient?

¶**Ex. 2.** As m becomes greater and greater, what becomes of the straight line $y = mx$?

223. Average gradient. Consider the case of an undulating road, or a curved graph. We cannot yet determine the gradient of a curve, as we have defined the term only for a straight line. But we can determine the *average* gradient over any length of the road, by calculating the gradient of a road of uniform slope that coincides with the actual road at the beginning and end of the length in question. Similarly, the average gradient of a curved graph between two points is the gradient of the chord joining these points.

Ex. 3. The height above sea-level of an undulating road is given in the following table:—

Horizontal distance in ft.	1000	2000	3000	4000	5000	6000	7000
Height above sea-level in ft.	65	90	73	124	159	128	66

Sketch the elevation of the road (take 1″ for 1000′ horizontally; 1″ for 100′ at right angles) and calculate the average gradient over each 1000′ length.

Keep your graph for use in Ex. 5 of § 225.

¶ For discussion in class.

Ex. 4. Find the average gradient of $y=x^2$ for every unit interval of x from $x=-4$ to $x=+4$.

224. Tangent to a curve. If a sheet of flat metal, cut with its edge in the shape of any curve, is laid flat on a table, and a straight edge pressed against the curved edge at any point, the edge in contact with the curve is a tangent to the curve at this point (fig. 35).

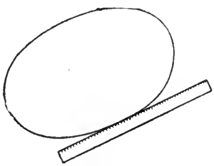

Fig. 35.

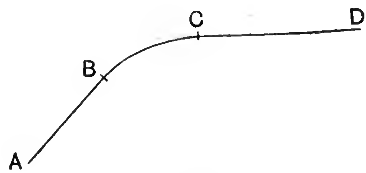

Fig. 36.

We get an equally good notion of what "tangent" means if we lay a piece of cotton along part of the curve, and pull it out straight from the point where it leaves the curve; the straight part of the cotton will be a tangent to the curve at this point. Again, the straight pieces AB, CD of the railway ABCD are tangents to the curve BC (fig. 36) at B and C respectively.

For mathematical purposes, it is found useful to treat the idea of tangent in the following way.

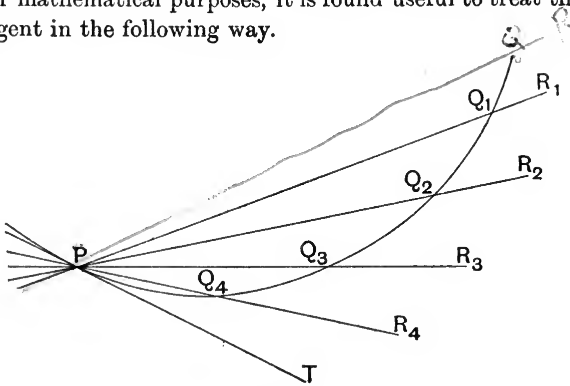

Fig. 37.

Through a point P of a curve (fig. 37) draw the chord PQ, meeting the curve again at Q: produce this chord indefinitely, past R. Now let the point Q move along the curve towards P, through the positions Q_1, Q_2, Q_3, etc., carrying the chord with it. The chord revolves about P, and the portion PQ becomes shorter and shorter. Let us attend, however, to PR, the chord produced.

The closer Q comes to P, the more nearly does PR give the true direction of the curve at P. As Q approaches closer and closer to P, so does PR approach closer and closer to a limiting position PT, which is called the tangent at P.

If Q were made to approach P from the other side, we should in general obtain the same limiting position. In a curve such as that shown in fig. 38,

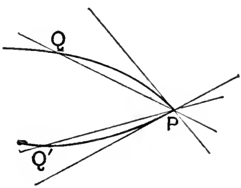

Fig. 38.

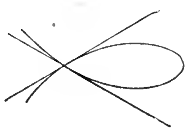

Fig. 39.

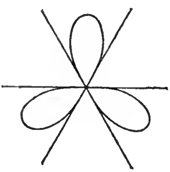

Fig. 40.

however, we should obtain different tangents according as Q approaches P from the one side or the other. Figs. 39 and 40 also show how a curve may have two or more tangents at a point: in these cases the same tangent is obtained as Q approaches P from one side or the other, provided that it approaches along the same "track."

Fig. 41 shows that the tangent at a point P may *cut* the curve at another point S.

Fig. 41.

225. Gradient of a curve. The gradient of a curve at any point is defined to be the gradient of the tangent at this point.

Ex. 5. Find the gradient of the road-graph of § 223 at 1000 ft., 4000 ft., and 6000 ft.

If we plot the graph of $y = x^2$ (see fig. 42), we see at once the gradient is positive on the right-hand side of OY, and negative on the left-hand side. At the origin the gradient is zero. Again, the gradient increases as x increases from 0.

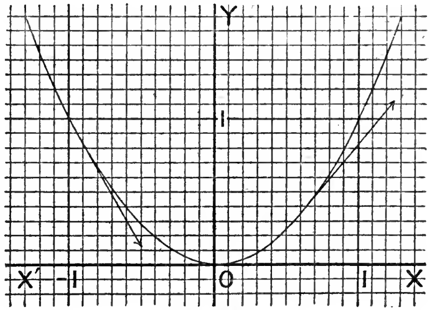

Fig. 42.

We might try to find the gradients at various points by actually drawing tangents, and finding the gradients of these tangents on squared paper. This would be quite easy, but as we have not explained the method of drawing the tangents accurately, the results would probably not be very good. But at any rate we could make some sort of table showing the gradient at various points of the curve, corresponding to various values of x. If we were able to do the drawing and measuring with perfect accuracy (which of course we never could do), we should find the following result:—

Gradient of $y = x^2$.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
Gradient	-10	-8	-6	-4	-2	0	2	4	6	8	10

Look at the table, and see that it contains all the features that are mentioned above. It suggests that there is a simple relation between the gradient and x: in fact that the gradient is $2x$.

226. Let us now deal with the problem arithmetically. To begin with, we will calculate the gradient at the point P where $x=1$. At this point $y=1$.

We have seen that the tangent PT (see fig. 37) is the limiting position to which a chord PQ approaches, as Q approaches closer and closer to P.

The gradient of PQ is the average gradient of the curve between the points P and Q. As Q approaches P, so does the average gradient between P and Q approach closer and closer to the gradient of the curve *at* P.

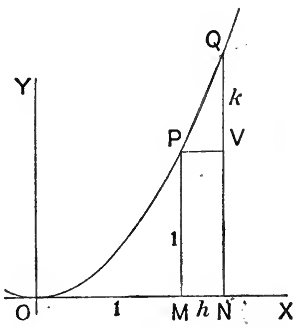

Fig. 43.

Let us therefore first calculate the gradient of the chord PQ. We have OM $=1$. Let MN $=\cdot 9$, so that the x of Q is $1\cdot 9$. Then the y of Q, namely NQ, $=1\cdot 9^2=3\cdot 61$, $\therefore$ VQ $=2\cdot 61$.

$$\therefore \text{ the gradient of PQ} = \frac{\text{VQ}}{\text{PV}} = \frac{2\cdot 61}{\cdot 9} = 2\cdot 9.$$

We will now take a succession of positions for Q, as it approaches P, the x's being $1\cdot 8$, $1\cdot 7$, $1\cdot 6$, ... $1\cdot 1$; the calculation of the gradients is as follows:—

ON	1·9	1·8	1·7	1·6	1·5	1·4	1·3	1·2	1·1
NQ=ON²	3·61	3·24	2·89	2·56	2·25	1·96	1·69	1·44	1·21
VQ	2·61	2·24	1·89	1·56	1·25	0·96	0·69	0·44	0·21
PV=MN	0·9	0·8	0·7	0·6	0·5	0·4	0·3	0·2	0·1
Gradient $=\frac{\text{VQ}}{\text{PV}}$	2·9	2·8	2·7	2·6	2·5	2·4	2·3	2·2	2·1

227. We cannot take the last step and put ON $= 1 \cdot 0$, for in that case the triangle QVP disappears. The successive gradients *suggest* however that the limiting value—the gradient of the tangent and the curve—is 2. We could test this further by taking another series of values of x, lying between $1 \cdot 1$ and $1 \cdot 0$, such as $1 \cdot 09$, $1 \cdot 08$, $1 \cdot 07$, ... But we shall show in § 228 that the limit really is 2.

*It will be well to spend a little time in examining the meaning of the word "limit."

Consider an expression such as $\frac{3h+2h^2}{h}$. This is a function of h, and its value can be found for any value of h except $h=0$. In fact, *unless* $h=0$, $\frac{3h+2h^2}{h} \equiv 3+2h$; and there is no difficulty. But if $h=0$, we have a fraction that looks like $\frac{0}{0}$. Now the process of division, as defined in arithmetic, does not contemplate 0 as a denominator. In fact, $\frac{0}{0}$ is an undefined and unmeaning combination; and when $h=0$, $\frac{3h+2h^2}{h}$ cannot be evaluated.

But though we cannot find a value for the function $\frac{3h+2h^2}{h}$ when $h=0$, we can find the limit to which the function tends as h tends towards 0; or more briefly, as $h \rightarrow 0$. For $\frac{3h+2h^2}{h} \equiv 3+2h$, whatever value h may have, however small, provided that $h \neq 0$. Impress upon your mind that 0 is entirely different from a small number, however small. Thus, a small object can be magnified to any required size; but no amount of magnification will make 0 anything but 0.

As $h \rightarrow 0$, $3+2h$ becomes more and more nearly equal to 3. We are inclined to say that the limit of $3+2h$ as $h \rightarrow 0$ *is* 3; but first note the following remark about the word "limit." Before we can assert that 3 is the limit of $3+2h$ as $h \rightarrow 0$, we must be prepared to satisfy a certain test. If one person (A) mentions any small number he cares to choose, another person (B) must be able by choosing h properly to make the difference between $3+2h$ and the alleged limit numerically less than this small number. There must be no collusion between A and B. Thus, A mentions $\frac{1}{100,000}$; can B make the difference between $3+2h$ and 3 numerically less than $\frac{1}{100,00}$?

* This may be omitted at first reading.

The difference is $2h$; and B can succeed by taking h somewhat less than $\frac{1}{200{,}000}$; say $h=\frac{1}{200{,}001}$. Whatever small number A chooses, B can always meet him. This being so, we may say that, as $h \to 0$, so $\frac{3h+h^2}{h}$ tends to the limit 3; or $\frac{3h+h^2}{h} \to 3$.

This may seem unnecessarily troublesome; why not put $h=0$ straight away? Because if we put $h=0$ in $\frac{3h+h^2}{h}$ we have nothing but the meaningless expression $\frac{0}{0}$.

228. In fig. 44, let $\mathrm{PV}=h$, $\mathrm{VQ}=k$. Then the coordinates of Q are $1+h$, $1+k$. But Q is on the curve $y=x^2$.

$$\begin{aligned}\therefore\ 1+k &= (1+h)^2\\ &= 1+2h+h^2.\\ \therefore\ k &= 2h+h^2.\end{aligned}$$

$$\therefore \text{ the gradient of PQ} = \frac{\mathrm{VQ}}{\mathrm{PV}} = \frac{k}{h} = \frac{2h+h^2}{h}.$$

Now, as Q approaches P, so $h \to 0$.

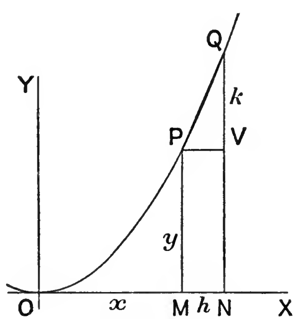

Fig. 44.

We have to find the limit to which $\frac{k}{h}$, or $\frac{2h+h^2}{h}$, tends as $h \to 0$. So long as $h \neq 0$, $\frac{2h+h^2}{h} = 2+h$; and as $h \to 0$, $2+h \to 2$.

Why is this true? If A says that he wants $2+h$ to differ from 2 by less than $\frac{1}{50,000}$, can B choose h so as to satisfy A? Clearly he has only to take $h=\frac{1}{50,001}$.

Recapitulating, $$k = 2h + h^2.$$

$$\therefore \frac{k}{h} = 2 + h, \text{ if } h \neq 0.$$

In the limit, as $h \to 0$, $\frac{k}{h} \to 2$. $\therefore$ the gradient of the curve at P $= 2$.

Ex. 6. Find, as above, the gradient of $y = x^2$ at points where $x = 2, 3, 4$.

229. Having found the gradient of $y = x^2$ at a number of particular points, we will now find the gradient at *any* point P.

In fig. 45, let OM $= x$; then MP $= y = x^2$.

Let PV $= h$, VQ $= k$.

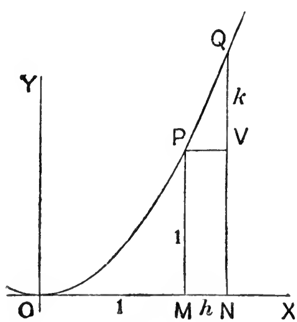

Fig. 45.

The coordinates of Q are $x + h$, $y + k$ and Q is on the curve.

$$\therefore y + k = (x + h)^2$$
$$= x^2 + 2xh + h^2.$$

But $$y = x^2.$$

$$\therefore\ k = 2xh + h^2.$$

$$\therefore \text{ the gradient of PQ} = \frac{k}{h} = 2x + h \text{ if } h \neq 0.$$

Now, as Q approaches P, $h \to 0$.

$$\therefore \text{ the gradient of the curve at P} = \text{limit of } (2x + h) \text{ as } h \to 0$$
$$= 2x.$$

It will be noticed that this is the function suggested by the table of gradients in § 225.

230. If we want to calculate the *ordinate*—i.e. the distance up to the curve—for any value of x, we work out the value of x^2.

If we want to calculate the *gradient* for any value of x, we work out the value of $2x$.

Thus x^2 may be described as the formula for the ordinate; $2x$ as the formula for the gradient.

Or, in other words, the function x^2 gives the ordinate, and the function $2x$ the gradient, for any value of x.

x^2 is the original function that defines the curve; $2x$ is called the **derived function** of x^2.

The process of finding the derived function of a given function is called **differentiation**: we are said to **differentiate** a function. The derived function of x^2 is written $\mathrm{D}x^2$; thus we have found that $\mathrm{D}x^2 = 2x$*. Note carefully that D is not a number or a multiplier; but simply an abbreviation for "the derived function of."

In the same way, the derived function of $2x^2$ is written $\mathrm{D}(2x^2)$; the derived function of $2x^2 - 3x + 5$ is written

$$\mathrm{D}(2x^2 - 3x + 5).$$

We may be considering a function of some variable other than x, say of t: thus $\mathrm{D}t^2 = 2t$.

If there is any ambiguity as to the variable with respect to

* Strictly, $\mathrm{D}x^2 \equiv 2x$.

which we are differentiating, we may use a suffix: thus the derived function of $2x^2+t$ with respect to x may be written $D_x(2x^2+t)$. If only one variable occurs in the function to be differentiated, the suffix is not needed.

EXERCISE XXIII. b.

(The following differentiations are to be performed from first principles as in § 229.)

1. Find the gradient of $y=2x^2$. Compare this with the gradient of $y=x^2$.

2. Find the derived function of $5x^2$, and compare it with that of x^2.

3. Differentiate $\frac{1}{2}x^2$, and compare the result with that of differentiating x^2.

4. Find the derived function of ax^2, a being a constant number.

5. Find $D(x^2+2)$; $D_x(x^2+c)$; $D_x(ax^2+c)$.

231. To differentiate a constant. This is the same as to find the gradient of $y=c$, where c is a constant. Now $y=c$ is a straight line parallel to the x axis and c units above it. Its gradient is therefore zero. Therefore the result of differentiating a constant is zero; in other words, the derived function of a constant is zero, i.e. $D_x c=0$.

232. To differentiate the function $mx+c$. This is equivalent to finding the gradient of $y=mx+c$. In § 222 this has been shown to be m.

$$\therefore\ D_x(mx+c)=m.$$

Or as follows. Let $y=mx+c$; and suppose that when x is

increased to $x+h$, y is increased to $y+k$. Since $(x+h, y+k)$ is a point on the graph

$$y+k=m(x+h)+c$$

$$=mx+mh+c.$$

$$\therefore\ k=mh.$$

$$\therefore\ \frac{k}{h}=m, \text{ a constant.}$$

$$\therefore\ \mathrm{D}_x(mx+c)=m.$$

Ex. 7. Read off the derived functions of

$$x, \quad 2x, \quad \tfrac{1}{2}x, \quad 3x-2, \quad 3x-5, \quad 4-3x, \quad a+bx.$$

233. We have now differentiated some very simple functions. Now suppose that we have succeeded in differentiating a function of x, what is the result of differentiating the product of that function and some constant factor, say c? For instance, we have differentiated x^2, what is the result of differentiating $3x^2$?

A function of x, i.e. an expression involving x, is denoted by $f(x)$. Note that $f(x)$ does not mean anything like a product of f and x, and that f is not a number; $f(x)$ is simply an abbreviation of "a function of x."

Ex. 8. If $f(x)=x^2+6x+7$, calculate $f(1)$, $f(2)$, $f(0)$, and $f(10)$.

We may state our problem thus:—If we know the result of differentiating $f(x)$, what is the result of differentiating $cf(x)$?

To differentiate $cf(x)$. If we compare the graphs of $f(x)$ and $cf(x)$, we see that the graph of $cf(x)$ can be obtained from that of $f(x)$ by multiplying all the perpendicular distances by c.

Fig. 46, p. 372 shows the graphs of x^2 and $2x^2$ on the same scale.

Suppose that when x is increased to $x+h$, $f(x)$ is increased to $f(x)+k$; then $cf(x)$ would be increased to $cf(x)+ck$.

Now the gradient of $f(x)$ is the limiting value of $\frac{k}{h}$ as $h \to 0$;

and the gradient of $c\,f(x)$ is the limiting value of $\frac{ck}{h}$ as $h \to 0$.

But $\frac{ck}{h}$ is always c times $\frac{k}{h}$. Thus the gradient of $cf(x)$ is c times that of $f(x)$; and the derived function of $cf(x)$ is c times the derived function of $f(x)$.

Shortly $$\mathsf{D}_x[cf(x)] = c\,.\,\mathsf{D}_x f(x).$$

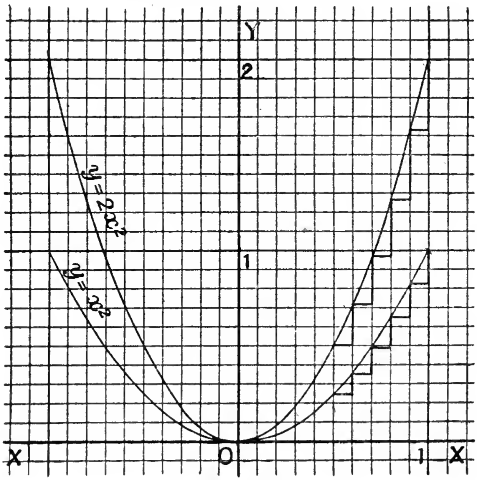

Fig. 46.

234. One more question must be answered before we can make much use of derived functions: what is the derived function of the sum of two functions?

To differentiate the sum of two functions. Take the example $5x^2 + 3x$. We can differentiate $5x^2$ and $3x$ separately; what is the derived function of their sum?

Let $y = 5x^2 + 3x$: and suppose that when x is increased to

$x+h$, y is increased to $y+k$. Since $(x+h, y+k)$ is a point on the graph,

$$\therefore\ y+k = 5(x+h)^2 + 3(x+h)$$
$$= 5x^2 + 10xh + 5h^2 + 3x + 3h.$$
$$\therefore\ k = 10xh + 5h^2 + 3h.$$
$$\therefore\ \frac{k}{h} = 10x + 3 + 5h, \text{ if } h \neq 0.$$
$$\therefore\ \frac{k}{h} \to 10x + 3 \text{ as } h \to 0.$$

Thus $$\mathrm{D}(5x^2 + 3x) = 10x + 3.$$

But $$\mathrm{D}(5x^2) = 10x,$$

and $$\mathrm{D}(3x) = 3.$$

Therefore, *in this case*, the derived function of the sum of two functions is simply the sum of the two derived functions.

This can be proved to be always true. In the same way the derived function of the difference of two functions is the difference of the two derived functions.

If we were to substitute for "difference" the word "product" or "quotient" the statement would be untrue. Thus $\mathrm{D}(x) = 1$; but $\mathrm{D}(x \times x)$ is not 1×1, but $2x$.

More generally, to write down the derived function of a polynomial, differentiate the terms individually, leaving the + and − signs untouched.

Example. Differentiate $2x^2 - 3x + \frac{1}{2}$.

$$\mathrm{D}(2x^2 - 3x + \tfrac{1}{2})$$
$$= \mathrm{D}(2x^2) - \mathrm{D}(3x) + \mathrm{D}(\tfrac{1}{2})$$
$$= 2 \times 2x - 3 \times 1 + 0$$
$$= 4x - 3.$$

With a little practice, such differentiations may be performed in one line.

EXERCISE XXIII. c. (oral.)

Read off the derived functions of the following functions:—

1. x^2+5.
2. x^2+3x.
3. $3x^2+x-1$.
4. $\frac{1}{2}x-\frac{3}{4}x^2$.
5. $3-\frac{4}{3}x^2+4x$.
6. ax^2+bx+c.
7. What is the geometrical meaning of the fact that $3x^2+x-1$ and $3x^2+x+2$ have the same derived function?

MAXIMA AND MINIMA.

235. Suppose that fig. 47 shows the graph of some function of x, which we will call $f(x)$. The points P_2, P_4 are called **maximum points** of the graph; the points P_1, P_3 are called

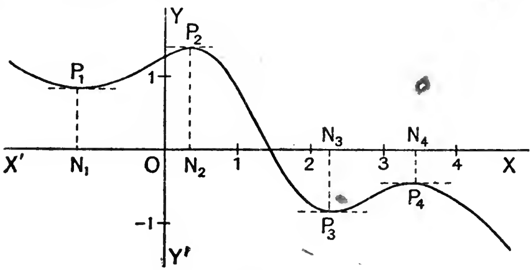

Fig. 47.

minimum points. The function has a maximum value N_2P_2 when $x=ON_2$; and a maximum value N_4P_4 when $x=ON_4$. Again, the function has a minimum value N_1P_1 when $x=ON_1$, and a minimum value N_3P_3 when $x=ON_3$.

Notice that in this case the minimum value at P_1 is greater than the maximum value at P_4. In fact, maximum and minimum values are not the same as greatest and least values, but are judged only with reference to the neighbouring points. Thus, if x is a little greater or smaller than ON_2, $f(x)$ will be somewhat smaller than N_2P. Again, if x is a little greater or smaller than ON_1, $f(x)$ will be somewhat greater than N_1P. Maxima are like mountain tops, and minima like valley bottoms.

We will assume that as x increases without limit beyond ON_4, $f(x)$ diminishes without limit and the graph continues to slope downwards. We could then find as large a range of values of x as we please that make $f(x)$ actually smaller than either of the minimum values. Similarly, if the curve always slopes upwards as we move to the left from P_1, we could find values of x that make $f(x)$ actually greater than either of the maximum values. A hilltop is higher than surrounding points, but there may be a higher mountain near. [illegible]

Just to the left of P_2 the gradient is positive; just to the right it is negative; *at* P_2 the gradient is zero. Does the same thing happen at P_4? We see that as P passes through a maximum towards the right, the gradient changes from + to − through zero.

How does the gradient change in passing through a minimum?

Points at which a function has a maximum or a minimum value are classed together as **turning points**, and the maximum and minimum values are called **turning values**. At every such point the function, after increasing, "turns" and begins to decrease; or vice versa.

To search for turning points of a function it is necessary to find values of x at which the gradient of the graph is zero, or the derived function of the function is zero.

236. Note on the terms "greater" and "less" as applied to negative numbers. -3 is said to be *greater* than -12. *Numerically*, -3 is less than -12; but from an algebraical point of view it is greater. The terms "greater" and "less" are always presumed to bear the algebraical meaning, unless the contrary is stated, i.e. unless we say "numerically greater" or "numerically less."

237. Before we can establish that a certain value of x, say $x=a$, gives a maximum or a minimum value of $f(x)$, we must make sure that the gradient changes sign as we pass from one side of $x=a$ to the other. The necessity for this is made clear by figs. 48 and 49. In each case, the gradient at P is zero, but

neither shows a maximum or a minimum. In fig. 48 what is the sign of the gradient to the left of P? what to the right? In fig. 49 what is the sign of the gradient to the left and to the right of P?

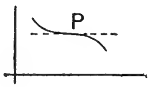

Fig. 48.

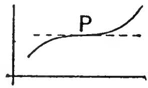

Fig. 49.

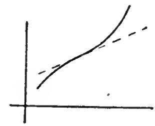

Fig. 50.

The curves in figs. 48 and 49 are said to have a **point of inflexion** at P. It is not, however, necessary that the tangent should be parallel to the x axis at a point of inflexion; thus fig. 50 shows a point of inflexion.

238. *Example.* To find the maxima or minima of

$$2x^2 - 3x + 1.$$

The derived function is $4x - 3$. This is zero when $x = \frac{3}{4}$; at this point the gradient of the graph is zero.

To ascertain whether this gives a maximum or minimum, we must find how the gradient behaves on each side of $x = \frac{3}{4}$. At $x = \frac{3}{4}$, $4x - 3 = 0$; if x is a little less than $\frac{3}{4}$, $4x$ is a little less than 3 and $4x - 3$ is negative. If x is a little greater than $\frac{3}{4}$, $4x$ is a little greater than 3, and $4x - 3$ is positive. Thus the gradient changes from $-$ to $+$ as x increases through the value $\frac{3}{4}$. This therefore gives a minimum.

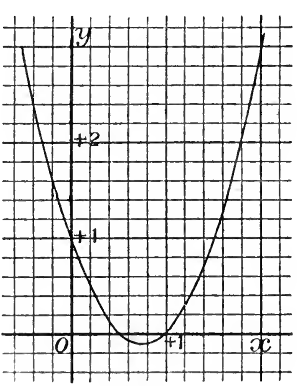

Fig. 51.

The value of the function at this point is

$$2 \times \tfrac{9}{16} - 3 \times \tfrac{3}{4} + 1 = -\tfrac{1}{8}.$$

This is shown in fig. 51.

Note that the function has no maximum values. It increases without limit as x increases or decreases without limit.

Example. What is the greatest rectangular area that can be enclosed by 200 hurdles each 2 yards long?

The total perimeter of the rectangle is 400 yards. Suppose that one side is x yards; the other side is then $(200-x)$ yards; the area is $x(200-x)$ sq. yds. It is required to find what value of x makes $x(200-x)$ a maximum.

$$\mathsf{D}[x(200-x)] = \mathsf{D}(200x - x^2) = 200 - 2x.$$

This is zero if $x = 100$, which gives a square of side 100 yards.

To ascertain whether this is a maximum or a minimum, consider the sign of the derived function $200-2x$ on either side of $x=100$. If x is a little less than 100, $200-2x$ is positive: if x is a little greater than 100, $200-2x$ is negative. $x = 100$ therefore makes the area a maximum.

The area has no minimum in a mathematical sense; what is the least value it can have?

EXERCISE XXIII. d.

1. For what value of x does the function $2 - 3x^2 + 6x$ attain a turning value? What is the turning value? Is it a maximum or a minimum?

2. Find the turning point of the graph $y = \frac{1}{2}x^2 - 2x - 7$.

3. The function $12x - 2 - 3x^2$ has a turning value; discuss whether it is a maximum or a minimum.

4. Investigate the turning values of the following expressions, i.e. find the value of x for which the expression has a turning value, and ascertain whether this turning value is a maximum or a minimum:—

(i) $2x(3-x)$,

(ii) $(x-1)(x-3)$,

(iii) $(x+1)^2 + (x+3)^2$.

5. Divide 10 into two parts such that (i) their product is a maximum, (ii) the sum of their squares is a minimum.

6. Given that the perimeter of a rectangle is $2p$, find for what shape the diagonal is least. (If the diagonal is a minimum, so also is its square.)

7. A rectangular channel pipe, open at the top, sides vertical, base horizontal, is bent up out of a long piece of thin sheet iron 12 inches broad, so that, x ins. being the base and y ins. the depth of the channel, $2y+x=12$ (see fig. 52). What is the area of the cross section of the pipe? For what value of x will it be greatest? (Eliminate y from the function that expresses the area.)

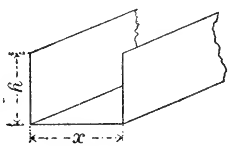

Fig. 52.

8. If a stone is projected with velocity u ft. per sec. in a direction making an angle 30° with the horizon, its height in feet after t seconds is $\frac{1}{2}ut-16t^2$. Find the value of t for which this is a maximum, and hence find the greatest height reached by the stone. (u is a constant for this problem.)

Also work out the particular case $u=64$, by substitution.

9. A and B start walking at noon towards a point O. A starts at P, 9 miles due South of O, and B starts at Q, 13 miles due West of O. A walks 4 miles an hour, and B 3 miles an hour. Show that if their distance apart after x hours is d miles, then $d^2=25x^2-150x+250$.

Hence find when their distance apart is least, and what this distance is. (Find when d^2 is least.)

10. The expense of running a goods train on a railway varies as the square of the number of trucks in it. The price charged for each truck is the same. If the receipts from a train of 40 trucks just pay the expenses of working it, show that a railway company will make the same profit on each train by

running trains made up of 17 trucks as they would by trains made up of 23 trucks.

Find also how many trucks should be put in each train in order that the profit may be as large as possible.

11. Draw a straight line OA, 4 inches long. A square PQRS has its centre at A and its sides, each of length $2x$, parallel and perpendicular to OA. PQ is the side nearest to O. Find for what value of x the length of OP is a maximum or a minimum, and whether the value you obtain corresponds to a maximum or to a minimum. (Take $OP^2 = y$, and find when y is a maximum or a minimum.)

12. A dynamo is in two parts whose weights are x lbs. and y lbs. The cost of the machine in pounds is $z = y + 4x$. The usefulness of the machine is proportional to $v = x^2 + 3xy$. If z is 10, express v in terms of x alone. For what value of x is v a maximum?

* SMALL INCREASES.

239. *Example.* A circular metal plate, of radius 2 ft., expands on a rise of temperature to radius 2·003 ft. Find approximately the increase of area.

At any instant let r ft. be the radius: A sq. ft. the area.

Then $$A = \pi r^2.$$

If A be plotted against r, $\dfrac{\text{increase of A}}{\text{increase of } r}$ = average gradient of graph. For approximate calculation, we may replace the average gradient over an interval by the gradient at any point in this interval; thus

$$\frac{\text{increase of A}}{\text{increase of } r} = D_r A \text{ approx.}$$
$$= 2\pi r \text{ approx.}$$

$\therefore$ increase of A is approximately $2\pi r \times$ increase of r
$$= 2\pi \times 2 \times \cdot 003$$
$$= \pi \times \cdot 012$$
$$= \cdot 037704, \text{ taking } \pi = 3{\cdot}142,$$

$\therefore$ increase of area $= \cdot 038$ sq. ft. to 2 sig. figs.

* May be omitted at first reading.

[The increase of area is exactly $\pi(2{\cdot}003^2 - 2^2)$

$$= \pi(2{\cdot}003 - 2)(2{\cdot}003 + 2)$$
$$= \pi \times {\cdot}003 \times 4{\cdot}003$$
$$= \pi \times {\cdot}012009 \text{ sq. ft.}$$

The error is $\pi({\cdot}012009 - {\cdot}012)$

$$= \pi \times {\cdot}000009$$
$$= {\cdot}000028\ldots \text{ sq. ft.}$$

The error is less than $\frac{1}{1000}$ of the result, or ·1 per cent.]

EXERCISE XXIII. e.

1. A cylinder of length 10 ins. and *diameter* $2\frac{3}{16}$ ins. is turned down to diameter $2\frac{1}{8}$ ins. Calculate approximately (to 2 sig. figs.) the diminution of volume: and find by how much per cent. the result is wrong (percentage to 1 sig. fig.).

2. If a copper bar is 1 foot long at 0° C., the length (l ft.) at θ° C. is given by the formula $l = 1 + a\theta + b\theta^2$, where

$$a = {\cdot}1607 \times 10^{-4}, \quad b = {\cdot}004 \times 10^{-6}.$$

Find an approximate value for the expansion between 10° C. and 11° C.

3. If the radius of a metal sphere increases 1 per cent. on a rise of temperature, prove that the surface increases approximately 2 per cent.

4. The distance of the visible horizon (x miles) as seen from a height of y feet above sea level is given by the formula $1{\cdot}5y = x^2$. To what height is it necessary to ascend in order to see 15 miles? Approximately how much higher must one climb to see an additional $\frac{1}{4}$ mile?

CHAPTER XXIV

RATE OF CHANGE

Rate of Change.

240. If a stone is let fall, it moves slowly at first and more and more rapidly as time goes on. If it were let fall from a high tower, or down a pit shaft, and the distances through which the stone falls in various intervals of time were measured, it would be found that distances and times are connected by a simple formula. If the stone falls s feet in t seconds from rest, i.e. from the instant when it is let fall, the formula $s = 16{\cdot}1t^2$ very nearly represents the observed facts. For simplicity of calculation, we will take $s = 16t^2$.

Every one has a fair notion of what is meant by moving faster or slower; everyday experience convinces one that after 2 seconds from rest a stone is falling faster than after 1 second from rest. But when it is asked, *exactly how fast* is the stone falling after 1 second, or after 2 seconds, the meaning of the question is not so obvious.

If I wish to know how fast a train in which I am travelling is moving, I can count the $\frac{1}{4}$ mile posts at the side of the track, and see how long the train takes to cover, say, 3 miles. If I then divide the number of miles by the number of minutes, I have as a result a velocity of so many miles per minute. If the train is moving with **uniform velocity** over this distance, i.e. neither slowing-down nor speeding-up, my result is good;

I have the velocity of the train in miles per minute. But suppose that the pace is increasing; what does my result stand for? All that I can say about my result is that it is the *average* velocity during the interval.

The **average velocity** of a moving body during a certain interval of time is measured by

$$\frac{\text{number of units of length traversed}}{\text{number of units of time in interval}}.$$

Ex. 1. Using the formula $s=16t^2$, find the average velocity of a falling stone, in feet per second, during the following intervals from rest:—

(i) 1 sec. to 4 secs. (iii) 1 sec. to 2 secs.

(ii) 1 sec. to 3 secs. (iv) 1 sec. to $1\frac{1}{2}$ secs.

Keep a note of your results for future reference.

241. If we plot the graph of $s=16t^2$, taking t along the horizontal axis, and s at right angles (in fact, taking t for x, and s for y), this graph is called a **space-time graph.** Of what shape is the space-time graph in this case?

The average velocity over an interval of time is measured by the ratio (change of s : change of t). This ratio may also be called the **average rate of change** of s. Notice now that this ratio is exactly the same as that which gives the average gradient of the graph. The average velocity, or the average rate of change of s, is represented graphically by the average gradient of the graph.

Ex. 2. Fig. 53 is the space-time graph of a train starting from rest. Find the average velocities (i) during the 1st minute, (ii) during the 2nd minute, (iii) during the first 2 minutes.

242. We will now enquire exactly what is meant by the velocity of the falling stone *at* a certain instant. We might perhaps be tempted to define it as the average velocity over a

small interval of time, beginning or ending at the specified instant. But would the average velocity be the same in both cases, and how small is the interval to be?

Now consider the following definition:—**The velocity at a certain instant is the limit of the average velocity during an interval beginning at the specified instant, as the interval tends to zero.**

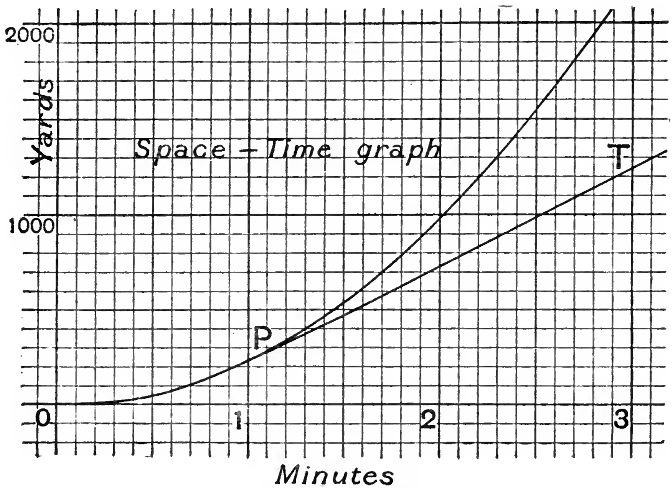

Fig. 53.

We have already seen that the limit of the average gradient is measured precisely by the derived function. In just the same way, the limit of the average velocity is measured by the derived function. In the present case, the velocity at the instant t is measured precisely by $D_t(16t^2)$ (remember that t now takes the place of x). What is $D_t(16t^2)$? What is the velocity after 1 second? How does this compare with the series of average velocities obtained in Ex. 1?

Just as the average velocity is represented graphically by the

average gradient in the space-time graph, so the velocity at a certain instant is represented graphically by the gradient of the space-time graph at the corresponding point.

Example. Taking the space-time graph of fig. 53, find the velocity at the end of the first minute.

The velocity is represented by the gradient of the graph at the end of 1 minute; i.e. by the gradient of the tangent at the point P. Draw the tangent PT. To find its gradient, choose the point T where the tangent cuts the 3-minute ordinate. The gradient is $\frac{1230-230}{2}=500$, $\therefore$ the velocity at the end of 1 minute $=500$ yards per minute.

243. We have discussed the case of a falling stone, the distance through which the stone has fallen being a function of the time. Now take another example, e.g. a sponge whose weight is increasing as it absorbs water. Suppose that its weight to begin with is a ounces; and that after t seconds the weight is $a+kt$ ounces. Precisely as before, we should define the rate at which the weight is growing at any instant as the limit of the average rate of increase during an interval beginning at the specified instant, as the interval tends to zero. Accordingly the rate of increase is $\mathrm{D}_t(a+kt)$ ounces per second $=k$ ounces per second.

EXERCISE XXIV. a.

1. The height in feet of a bamboo is observed every day at noon; the observations are:—

Day	1	2	3	4	5	6	7	8	10	12	14
Height	0	2·4	4·4	6·3	8	9·5	10·8	12·1	14·2	16	17·2

Draw a curve to exhibit this, and obtain from it the approximate rate at which the bamboo is growing, in feet per 24 hours, at noon on the 6th day.

2. A train, starting from rest, moves a distance of 800 yards in 50 seconds, and the table below gives corresponding values of distance x in yards and time t in seconds. Plot the space-time graph, and deduce from it the velocities at 0, 10, 20, 30, 35, 40, 50 seconds, hence plot the velocity-time graph :—

t	0	5	10	15	20	25	30	35	40	45	50
x	0	10	40	80	140	220	340	520	690	780	800

3. The following table gives observations of the Centigrade temperature of some water in a vessel.

The temperature taken at intervals of 5 minutes was :—

$$80°, 67°, 56°, 47°, 40°, 35°, 31°.$$

Plot the temperatures against the times and join by a smooth curve.

Estimate, as nearly as you can, the rate at which the temperature was falling at the instant when the temperature was 60°, and when it was 50°, and when it was 40°.

4. The amount of water in a tank at various times after opening a tap is given in the following table :—

Time in seconds	0	5	10	15	20	30	40	50	55	60	70
Amount of water in gallons	64	56·2	49	42·3	36	25	16	9	6·25	4	1

Plot these quantities and deduce from the curve the rate at which water was leaving the tank at the end of 15 seconds and 55 seconds.

5. A point passes over 0 feet in 0 secs. Could its speed be 30 feet per sec.? Could it be 1000 feet per sec.? Could the point be at rest? Give reasons for your answer.

6. In the case $s=16t^2$, sketch a graph to show how the velocity depends on the time. Also put into words the relationship between (i) distance and time, (ii) velocity and time (say "distance varies as —," "velocity varies as —").

7. If $s = 4 - 3t + t^2$, how does the velocity depend on the time? At what instant is the velocity zero? When is it positive and when is it negative? When has the distance from the origin a turning value? Is this a maximum or a minimum?

8. If $s = 2t^2 - 4t + 5$, when is the velocity zero? positive? negative? When is s a minimum?

9. A man 6 ft. high is walking at v ft. per sec. directly away from a lamp-post, the light being 10 ft. above the ground. Find (i) the velocity of the end of his shadow, (ii) the rate at which his shadow is growing.

(At t secs. after he passes the lamp-post, he is vt ft. from the post. Let distance of end of shadow from post be x ft., length of shadow y ft. Prove that $\frac{x}{10} = \frac{y}{6} = \frac{vt}{4}$; hence that velocity of end of shadow : rate of growth of shadow : velocity of man $= 5 : 3 : 2$.)

Ratio of Rates of Change.

244. Suppose that a beam of light is thrown through a square aperture on to a screen, making a square patch of light on the screen. The distance between the lantern and screen is varied, the size of the square patch of light varying accordingly. When the side of the square is x ins. and is increasing at the rate of 2 inches per second, at what rate is the area increasing?

Let x ins. be the side of the square, y sq. ins. the area. Then $y = x^2$.

We are told that $D_t x = 2$; what is $D_t y$?

Suppose that as t increases by τ, so x increases by h, and y increases by k.

Then $$D_t x = \text{limit of } \frac{h}{\tau} \text{ as } \tau \to 0,$$

and $$D_t y = \text{limit of } \frac{k}{\tau} \text{ as } \tau \to 0.$$

But $\dfrac{k}{\tau}\Big/\dfrac{h}{\tau} = k/h$; and we shall assume that

$$\frac{\text{the limit of } \dfrac{k}{\tau}}{\text{the limit of } \dfrac{h}{\tau}} = \text{the limit of } k/h.$$

But the limit of $k/h = \mathrm{D}_x y$,

$$\therefore\ \mathrm{D}_t y/\mathrm{D}_t x = \mathrm{D}_x y,$$

$$\therefore\ \mathrm{D}_t y = \mathrm{D}_x y \times \mathrm{D}_t x.$$

Now $$\mathrm{D}_x y = 2x \text{ and } \mathrm{D}_t x = 2,$$

$$\therefore\ \mathrm{D}_t y = 4x.$$

$\therefore$ the area is increasing at the rate of $4x$ sq. ins. per sec. For example, if the side of the square is 12 ins., the area is increasing at the rate of 48 sq. ins. per sec.

Note that while the side increases uniformly, the area increases faster and faster.

In general, if y is a function of x,

$$\frac{\text{rate of change of } y}{\text{rate of change of } x} = \mathrm{D}_x y.$$

If y be plotted against x, the ratio of the rates is given by the gradient of the graph. Accordingly we may look upon the gradient as giving the rate of change of y compared with that of x, which is sometimes called the x-rate of change of y.

EXERCISE XXIV. b.

1. The radius of a circle increases uniformly at the rate of ·1 inch per second. At what rate is the area increasing when the radius is 1 foot? $2.4\pi = 7.54''/s$

2. Find where the x and the y of the curve $y = 5x - x^2$ are increasing at the same rate; and also where y is increasing twice as fast as x.

3. A spherical soap bubble is expanding, the radius increasing at the rate of a inches per second. At what rate is the surface of the soap bubble increasing when the radius is 6 ins.? 9 ins.? (Surface $= 4\pi r^2$.)

4. A V-shaped drinking trough is 3 ft. long, and its vertical cross-section is a right-angled triangle. Prove that when the depth of the water is x ins., the volume of water in the trough is $36x^2$ cu. ins. If water is flowing in at the rate of 1000 cu. ins. per minute, at what rate is the surface rising when the depth is 6 ins.? 10 ins.?

5. A pianola record, in the form of a long strip of paper a ins. wide, is being wound on to a cylinder a ins. long at the rate of b cubic inches per second. At what rate is the radius of the roll increasing when the radius is x ins.?

Motion in Two Dimensions.

245. Imagine a point to be moving along a plane curve. If we draw the axes of x and y, we find that at any instant the x and y of the point have certain values. Let the instant be specified by saying that it is t seconds after a fixed instant of time. Then at 5 seconds after this instant, x and y will have certain values; at 10 seconds after this instant, x and y will have certain other values; etc. In fact x and y are functions of t. In order to grasp this idea, the reader should work the following exercise.

EXERCISE XXIV. c.

1. Plot the path of a point whose positions at the specified times are given by the following table:—

t	0	1	2	3	4	5	6	7	8	9	10	11
x	3	2·60	1·50	0	−1·50	−2·60	−3	−2·60	−1·50	0	1·50	2·60
y	0	1	1·73	2	1·73	1	0	−1	−1·73	−2	−1·73	−1

2. A cricket ball leaves the bat with a velocity of 100 ft. per sec., at a certain angle of elevation. If the resistance of the air be neglected, the horizontal distance from the starting point (x feet) and the height above the ground (y feet) after t seconds are given by the equations: $x = 80t$, $y = 60t - 16t^2$. Make a table showing the values of x and y at intervals of $\frac{1}{2}$ second from $t = 0$ till the ball reaches the ground again; hence plot the path of the ball. By solving the equation $y = 0$, find the value of t when the ball reaches the ground. Hence calculate the horizontal length of the hit.

246. Suppose that the cricket ball referred to in the preceding exercise had been hit in the tropics at an instant when the sun was vertically overhead. The shadow of the cricket ball would then be vertically below the ball, and at a distance of x ft. from the starting point. The velocity of this shadow is measured by the rate of increase of x. The actual motion of the ball at any instant is neither horizontal nor vertical, but inclined to the ground. But if we consider simply the rate at which x is increasing, we call this the **horizontal velocity** of the ball in feet per second; this is the same as the velocity of the shadow under a vertical sun.

Similarly, the rate at which y is increasing is called the **vertical velocity** of the ball in feet per second. If the sun was just setting and the ball hit directly towards the sun, what would be the path of the ball's shadow thrown on a vertical wall? What relation would there be between the velocity of this shadow and the vertical velocity of the ball?

The actual velocity of the ball can be calculated easily if we know its horizontal and vertical velocities: but it would lead us too far from our subject if we were to explain this.

EXERCISE XXIV. d.

1. In the cricket ball example (see Ex. XXIV. c. 2 and § 246)

(i) find the horizontal velocity, i.e. D $(80t)$. What do you notice about the value of this at different instants of time?

(ii) find the vertical velocity. What is the value of this at the start? How does it vary as t increases?

(iii) when is the vertical velocity zero, and how is the ball moving at this instant? How high is the ball at this instant? Find the maximum height of the ball.

(iv) what is the vertical velocity of the ball at the instant when it reaches the ground again? How does this compare with the vertical velocity at the start?

2. Prove that the equation of the ball's path through the air is $y=\frac{3}{4}x-\frac{x^2}{400}$.

3. A projectile is moving along a path whose equation is

$$y=\frac{x}{20}-\frac{x^2}{62500},$$

the unit of length being 1 foot and the gun being at the origin; its horizontal velocity is uniformly 1000 ft. per sec.: find its vertical velocity as a function of x (see § 244). Hence find at what horizontal distance the projectile ceases to rise.

ACCELERATION.

247. We have seen that, in the case of a point moving along a straight line, the rate of increase of the distance from the origin is the velocity. Now the velocity may either be uniform or changing, and in the case of a changing velocity the question arises—at what rate is the velocity changing?

If a point is moving along a straight line, the rate of change of velocity is called **acceleration.** If the velocity is increasing, the acceleration is positive; if the velocity is decreasing, the acceleration is negative, in which case its numerical value is often spoken of as **retardation.** Thus we may speak of the acceleration of a train when it is starting, and the retardation when the brakes are applied. The word "acceleration" is used in everyday language, e.g. when it is remarked that the acceleration (and retardation) of an electric train is greater than that of a steam train: the greater the acceleration, the greater the difficulty in standing up when the train is starting (or stopping).

248. The unit of velocity is 1 foot per second; 1 mile per hour; etc., according to the units of length and time adopted. If a point, in t seconds, increases its velocity from u ft. per sec. to v ft. per sec., it has gained $(v-u)$ units of velocity during t seconds. Its **average acceleration** during this interval is defined to be $\frac{v-u}{t}$ units of velocity per second. As the unit of velocity is 1 foot per second, the average acceleration is $\frac{v-u}{t}$ ft. per sec. per sec. Notice that the phrase "per second" occurs twice.

If we found that a train increased its velocity from u miles per hour to v miles per hour, in the course of t minutes, the average acceleration would be $\frac{v-u}{t}$ miles per hour per minute. The double mention of time may be made more intelligible if the acceleration is written $\frac{v-u}{t}$ (miles per hour) per minute; the bracketed (miles per hour) defining the unit of velocity used. In practice the brackets are never used.

249. Just as the velocity at an instant is derived from the average velocity during an interval by proceeding to the limit as the interval tends to zero, so the acceleration at an instant is

derived from the average acceleration during an interval by proceeding to the limit as the interval tends to zero.

And just as the velocity at an instant is calculated by taking the derived function of the distance, so the acceleration is calculated by taking the derived function of the velocity. The acceleration at an instant is represented graphically by the gradient of the velocity-time graph at the corresponding point.

Consider the case of the falling stone. Let the distance fallen in t seconds be s feet; the velocity attained in t seconds be v ft. per sec.; the acceleration at t seconds be f ft. per sec. per sec. Then

$$s = 16t^2,$$

$$v = \mathsf{D}_t s = \mathsf{D}(16t^2) = 32t,$$

$$f = \mathsf{D}_t v = \mathsf{D}(32t) = 32.$$

We see therefore that in the particular case of a falling stone the acceleration is constant. (Actually this is not quite true, as the resistance of the air comes into play; in fact the formula $s = 16t^2$ is not quite true for a stone falling *through the air.*)

EXERCISE XXIV. e.

1. The speed of a train is increased as follows:—

Time in seconds from start	0	4	8	12	16	20	24
Velocity in feet per second	0	1·4	3·6	6·3	9·4	12·4	13

Draw the velocity-time graph, and from it determine the acceleration of the train 12 secs. after the start.

2. A body starts from rest and moves in a straight line so that its velocity after t seconds is given by $v = 6t - 4t^2$. Find (i) the acceleration after 4 seconds, (ii) the average acceleration during the 4th second.

3. A body moves in a straight line; and s, the number of feet from a fixed point after t seconds, is given by the formula

$$s = \frac{3}{2} - 2t + \frac{1}{2}t^2.$$

(i) How far is the body from the fixed point at the instant from which time is reckoned?

(ii) What is the velocity after t seconds?

(iii) Sketch the velocity-time graph.

(iv) What is the velocity when $t = 0$? What is the meaning of its negative sign?

(v) When does the body begin to move in a positive direction?

(vi) When does the body pass through the origin? Account for the double answer.

(vii) What is the acceleration at the start? after 5 seconds?

CHAPTER XXV

FURTHER EXAMPLES OF DERIVED FUNCTIONS

250. Notation. In differentiating x^2, we proceeded as follows. "Let $y = x^2$; and suppose that when x is increased to $x + h$, y is increased to $y + k$,

$$\therefore \; y + k = (x + h)^2, \text{ etc.''}$$

It is now convenient to introduce the following notation. Instead of h write Δx (this is read "delta x"); instead of k write Δy. Δx means "the increment of x"; Δy means "the increment of y." The symbol Δ is not a quantity, or a multiplier of x or y: it has no meaning apart from the x or y to which it refers. Thus $\frac{\Delta y}{\Delta x}$ means the ratio of the increment of y to the increment of x; the Δ's cannot be cancelled. In fact, $\frac{\Delta y}{\Delta x}$ is what we have hitherto written $\frac{k}{h}$, and measures the average gradient of the graph (whatever the graph may be) over the interval between x and $x + \Delta x$.

The limit of $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$ is the gradient of the graph at the point given by x; it is what we have hitherto called $\mathrm{D}_x y$. But the more usual notation for this limit is $\frac{dy}{dx}$: thus

$$\text{the limit of } \frac{\Delta y}{\Delta x} \text{ as } \Delta x \to 0 \text{ is } \frac{dy}{dx}.$$

The reader should note carefully that $\dfrac{dy}{dx}$ is not a ratio or a quotient; the symbols dy, dx written separately have no meaning. For instance, the limit of Δx is not dx, the limit of Δy is not dy. The symbol $\dfrac{dy}{dx}$ means neither more nor less than $\mathrm{D}_x y$—the derived function of y with respect to x.

The derived function of x^2 with respect to x may be written $\dfrac{d(x^2)}{dx}$; or more usually $\dfrac{d}{dx}(x^2)$. The derived function of $2x^2 - x$ with respect to x is written $\dfrac{d}{dx}(2x^2 - x)$; and so forth.

251. To differentiate x^3. Let $y = x^3$; and suppose that when x is increased to $x + \Delta x$, y is increased to $y + \Delta y$.

Then
$$y + \Delta y = (x + \Delta x)^3$$
$$= x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3,$$
$$\therefore\ \Delta y = \cancel{x^3} + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3, -\cancel{x^3}$$
$$\therefore\ \frac{\Delta y}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2, \text{ if } \Delta x \neq 0.$$

Therefore, as $\Delta x \to 0$, $\dfrac{\Delta y}{\Delta x} \to 3x^2$,
$$\therefore\ \frac{dy}{dx} = 3x^2, \text{ or } \frac{d}{dx}(x^3) = 3x^2.$$

252. To differentiate $\dfrac{1}{x}$. Let $y = \dfrac{1}{x}$; and suppose that when x is increased to $x + \Delta x$, y is increased to $y + \Delta y$.

Then
$$y + \Delta y = \frac{1}{x + \Delta x},$$
$$\therefore\ \Delta y = \frac{1}{x + \Delta x} - \frac{1}{x}$$
$$= \frac{x - (x + \Delta x)}{x(x + \Delta x)}$$
$$= \frac{-\Delta x}{x(x + \Delta x)},$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = -\frac{1}{x^2 + x\Delta x}, \quad \text{if } \Delta x \neq 0.$$

Therefore, as $\Delta x \to 0$, $\dfrac{\Delta y}{\Delta x} \to -\dfrac{1}{x^2}$.

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{x^2}, \quad \text{or} \quad \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

EXERCISE XXV. a.

1. Show that the gradient of $y = x^3$ is always positive. What does this mean?

2. The point (2, 8) is on $y = x^3$; what is the gradient at this point?

3. What is the gradient of $y = x^3$ (i) where $x = -\frac{1}{2}$, (ii) where $y = -27$?

4. Where does $y = x^3$ slope at $45°$?

5. Show that the gradient of $y = \dfrac{1}{x}$ is always negative. What does this mean?

6. At what point does the curve $y = \dfrac{1}{x}$ slope at an angle of $45°$?

7. Find the slopes of $y = \dfrac{1}{x}$ and $y = x^3$ at the points where the two curves cut. Mark on a sketch the points and the tangents at these points.

253. Recapitulation of results. We have now differentiated the following powers of x; namely x^3, x^2, x, $\dfrac{1}{x}$. We

may also regard 1 as a power of x, since $x^0 = 1$. The results obtained are as follows:—

$$\frac{d}{dx}(x^3) = 3x^2,$$

$$\frac{d}{dx}(x^2) = 2x \text{ or } 2x^1,$$

$$\frac{d}{dx}(x^1) = 1 \text{ or } 1 \, . \, x^0,$$

$$\frac{d}{dx}(x^0) = 0,$$

$$\frac{d}{dx}(x^{-1}) = -\frac{1}{x^2} \text{ or } (-1)\, x^{-2}.$$

In more advanced books, the method is shown of differentiating *any* power of x. Can you, by examining the above table, suggest any general law for the derived function of x^n?

254. *Example.* Investigate the turning values of

$$2x^3 + 3x^2 - 36x + 10.$$

Let y denote the given function of x.

Then
$$\begin{aligned} \frac{dy}{dx} &= 6x^2 + 6x - 36 \\ &= 6\,(x^2 + x - 6) \\ &= 6\,(x-2)(x+3). \end{aligned}$$

The function has turning values where $\frac{dy}{dx} = 0$; i.e. at $x = 2$ and -3.

To find whether these values are maxima or minima, we must examine the sign of $\frac{dy}{dx}$ near these points.

If $x > 2$, $6\,(x-2)(x+3)$ is + ve.

If $2 > x > -3$, $6\,(x-2)(x+3)$ is − ve.

If $x < -3$, $6\,(x-2)(x+3)$ is + ve.

Therefore, when x is just less than 2, $\frac{dy}{dx}$ is $-$ ve; and when x is just greater than 2, $\frac{dy}{dx}$ is $+$ ve. $\therefore$ $x = 2$ makes y a minimum. At this point, $y = 16 + 12 - 72 + 10 = -34$.

Again, when x is just less than -3, $\frac{dy}{dx}$ is $+$ ve; and when x is just greater than -3, $\frac{dy}{dx}$ is $-$ ve. $\therefore$ $x = -3$ makes y a maximum. At this point, $y = -54 + 27 + 108 + 10 = 91$.

EXERCISE XXV. b.

1. Write down the value of $\frac{d}{dx}\left(\frac{1}{3}x^3 - 2x^2 - \frac{3}{x}\right)$.

2. Differentiate :—

(i) $10 - \frac{2}{3}x^3 + \frac{1}{2x} - \frac{1}{2}x^2$,

(ii) $3x - 5 + 4x^3 - 6x^2 - \frac{3}{x}$,

(iii) $-\frac{3}{5x} + \frac{1}{6}x^3 - 7$.

3. Write down the derived functions of

(i) $2x(3 + x^2)$ (first remove brackets),

(ii) $\frac{1}{x}(3 + 4x + 5x^2 + 6x^3)$,

(iii) $\frac{4x^5 - 3x}{x^2} - \frac{2x^3 - 4x}{x}$,

(iv) $x(2 + 3x)^2$.

4. Find the gradient of the curve $xy = a^2$ at the point where $x = 2a$.

5. If $R = 2\theta(a - b\theta^2)$, find $\dfrac{dR}{d\theta}$ where a, b are constant.

6. For what values of x are the following expressions positive? for what values are they negative?

(i) $(x-5)(x-1)$, (iv) $(3+x)(5-x)$,

(ii) $(x-1)(x-2)(x-3)$, (v) $(x-1)^2(3-x)$,

(iii) $(2-x)(x-4)$, (vi) $(x^2-1)(x^2-4)$.

7. Show that the expression $x^3 - 9x^2 + 24x - 7$ decreases in value as x increases from 2 to 4, and after that increases as x increases.

8. Investigate the maxima and minima of the following functions :—

(i) $x^3 - 3x$, (iii) $4x + \dfrac{1}{x}$,

(ii) $\dfrac{x^3}{3} - 2x^2 + 3x$, (iv) $\dfrac{x^3}{3} - 5x - \dfrac{4}{x}$.

9. In the expression

$$E = \frac{3\left(x^2 + \frac{1}{2}\right)}{4x}$$

it is known that a maximum or minimum value of E lies between $x = 0$ and $x = 1$. Find the values of E and of x at this point, and state whether (and why) E is then a maximum or a minimum.

10. When is the sum of a number and its reciprocal a minimum? a maximum?

EXERCISE XXV. c.

1. A rectangular solid has a square base. Its combined length, breadth and depth add up to 14 feet. If the length and breadth are each x feet, what is the depth? Write down in terms of x an expression for the volume. Find, within a tenth, the value of x for which this volume is greatest. What is this maximum volume?

2. A rectangular sheet of metal is 12 inches by 8 inches. Four equal squares, side x inches, are removed from each corner. The edges left are then turned up to be perpendicular to the base, and so form a box. Show that it holds $(4x^3 - 40x^2 + 96x)$ cubic inches.

Find the value of x that makes this volume greatest.

3. A closed rectangular box is to be made to contain 10 cubic feet. It is to be 3 feet long. If x feet be the breadth and y sq. ft. the area of the internal surface, show that

$$y = \frac{20}{3} + 6x + \frac{20}{x}.$$

Find the value of x that gives a minimum value for the surface.

X 4. An open tank with a square base and vertical sides is to have a capacity of 4000 cubic feet. Find the dimensions so that the cost of lining it with lead may be a minimum.

5. A circular cistern without a top is to be constructed to hold 800 cubic feet. Prove that the number of square feet (y) in the surface is given by the equation

$$y = \pi x^2 + \frac{1600}{x},$$

where x is the radius of the base of the cistern in feet.

Hence find the least value of y.

6. A match-box, made of wood of negligible thickness, is c inches long and consists of a box, open at the top, and a cover, open at both ends, into which the box slides. Take x inches as the breadth and y inches as the depth of the box and cover and suppose that the volume of the box is given equal to a^2c cu. ins. Express the area of wood required for the box and cover in terms of x, a, c; find the ratio of x to y so that this may be a minimum, when a and c are given.

7. An open rectangular tank whose depth is y feet and base a square of side x feet [inside measurements] is to have an inside capacity a^3 cubic feet. It is made of two pieces of metal riveted at the four sides of the base, and along one of the vertical sides. If the cost of riveting is £b per foot length of riveted seam measured inside, find the proportions of the tank for which the cost of the riveting will be a minimum. Give a common sense reason as to why this cost is a minimum and not a maximum.

8. The strength of a rectangular beam of given length varies as xy^2, where x ins. is the breadth and y ins. the depth. Find the depth and breadth of the strongest rectangular beam of perimeter 3 feet.

9. Under certain conditions, the strength of a rectangular beam of given length varies as the product of the breadth and the square of the depth. Given a cylindrical tree trunk of diameter d inches, what is the ratio of depth to breadth for the strongest rectangular beam that can be cut from it? (Take x inches as the breadth of the beam.)

10. A cylindrical vessel is to be made out of 100 square inches of sheet tin. It has no lid. If the radius of the base is x inches, show that the volume is $\left(50x - \frac{\pi}{2}x^3\right)$ cubic inches. What is the greatest volume of the vessel?

11. A piece of wire of length 6 feet is to be cut into six portions, two equal of one length and four equal of another. The two former are each bent into the form of a circle and these are held in parallel planes, and fastened together by the four remaining pieces which are perpendicular to the planes of the circles. The whole thus forms a model of a cylinder. Calculate the lengths into which the wire must be divided so as to produce a cylinder of maximum volume.

12. Given the area of a rectangle, find when the perimeter is least.

13. A cylinder is inscribed in a given sphere, find when the cylinder has maximum volume. (Let the height of the cylinder be $2x$ units.)

14. A wooden cylinder is to be covered with a given area of gold leaf: prove that the volume of the cylinder is greatest when the height is equal to the diameter of the base. [Prove that $r(r+h)=$constant, say k^2. Express the volume as a function of r.]

15. When a certain ship is steaming at v knots the coal consumption per hour is known to be $(0{\cdot}25 + {\cdot}002v^3)$ tons. The constant expenses for wages, upkeep, etc., work out at the cost of 1·5 tons of coal per hour. Find the cost of steaming 1000 sea-miles at v knots, assuming the coal to cost £c per ton. Now determine v so that this cost may be least.

16. On a certain steamer the cost of coal per mile is found to be directly proportional to the square of the speed maintained, being 2*s.* 6*d.* per mile when the speed is 30 miles per hour; the other expenses being £1. 16*s.* 0*d.* per hour at all speeds. Express in pounds the expense of a journey of 100 miles at x miles an hour. Hence find the most economical speed and the expense at that speed.

17. In running an express train from London to Edinburgh the cost is £$\left(a\mathsf{V}+\dfrac{b}{\mathsf{V}}\right)$, where V miles per hour is the average velocity of the train, and a, b are constants. When the train travels at the rate of 30 miles an hour the cost is £75; when it runs at 40 miles an hour the cost is £65. Find the values of a and b. Hence find the most economical speed and the expense at that speed.

18. The Parcel Post regulations require that the sum of the length and girth of a parcel shall not exceed 6 ft. Girth means measurement round. If it is desired to send the greatest possible volume, find the length of the parcel (i) if it is a prism with square section (let side of section $=x$ ft.), (ii) if it is a cylinder. Also find the volume in each case.

19. Soreau's formulae for the supporting thrust V and the horizontal thrust H of the air on a plane surface making a small angle α with the direction of motion are

$$\mathrm{H} = kv^2(a\,\alpha^2 + b),$$

$$\mathrm{V} = kv^2\,\alpha,$$

where v is the velocity of the plane and k, a, b are constants. For what value of α is the ratio $\dfrac{\mathrm{H}}{\mathrm{V}}$ a minimum?

20. It is found that the cost of running a certain steamer a certain definite distance, at an average speed of V knots, is proportional to $\mathrm{V} + \dfrac{\mathrm{V}^3}{100} + \dfrac{300}{\mathrm{V}}$, the first two terms representing the cost of power and the third term the costs, such as wages, which are directly proportional to the time occupied. What is the most economical speed?

EXERCISE XXV. d.

1. A body moves in a straight line in such a way that its distance (s feet) from a fixed point O in the line at the end of t seconds is given by $s = 11 + 2t^3$. Find its distances from O at the end of 4, $4+h$, $4-h$ seconds. Hence find the average speeds

(i) during the interval of h seconds immediately following the end of the 4th second;

(ii) during the interval of h seconds immediately preceding the end of the 4th second;

(iii) during the interval of $2h$ seconds, which is the sum of these two intervals.

Show that by making h small enough, each of these average speeds can be brought as near as we like to 96 ft./sec. What is this speed of 96 ft./sec. called?

2. A particle moves in a straight line in such a way that its distance from a fixed point O in the line at the end of t seconds is $(17 + 5t + 3t^3)$ feet. Find (i) its distance from O, (ii) its speed, (iii) its acceleration, at the end of 4 seconds.

3. The distance (s feet) that a point P, which moves on a straight path, has passed over in t seconds is given by

$$s = 5 + 10t - 4t^3.$$

(i) What is its velocity when $t = 1$?

(ii) When is it instantaneously at rest?

4. If $s = t^3 + 7t$ (s being measured in feet and t in seconds), find

(i) average speed between $t = 2$ and $t = 3$,

(ii) arithmetic mean of the speeds at the instants $t = 2$ and $t = 3$,

(iii) speed at the instant $t = 2\frac{1}{2}$.

5. The distance (s feet) passed over by a body in t seconds is given by

$$s = 4t - 5t^2 + 2t^3.$$

Find the velocity and acceleration after 3 seconds.

6. If a body be moving in a straight line and its distance (s feet) from a fixed point in the line after t seconds is given by $s = 5 + 2t + t^3$, find

(i) the average speed during the 5th second,

(ii) the speed at the end of $4\frac{1}{2}$ seconds,

(iii) the acceleration at the end of $4\frac{1}{2}$ seconds.

7. A body moves in a straight line so that its distance (s feet) from a fixed point in the line at the end of t seconds is given by

$$s = 5 - 3t + 7t^2 - t^3.$$

Find (i) its distance from the fixed point, (ii) its speed, (iii) its acceleration, at the end of 4 seconds.

8. If $s = a + bt + ct^2 + dt^3$ with usual notation, and a, b, c, d are constants, find

(i) average velocity between the end of 2nd and 4th second,

(ii) velocity at the end of 3 seconds,

(iii) acceleration at the end of 3 seconds.

9. The radius of a sphere is increasing at the rate of 0·02 inch per minute; at a certain instant the radius is 10 inches; find at what rate per minute the volume of the sphere is increasing at that instant.

10. The volume of a sphere is increasing uniformly at the rate of 1 cubic inch per second. Find at what rate the radius is growing (i) when the radius is 3 inches, (ii) when the radius is 10 inches.

11. A hemispherical bowl of radius 1 foot is partly filled with water. If the depth of water is x ins., and y sq. ins. the area of the water-surface, express y in terms of x.

If water is poured in at such a rate that x increases uniformly at 1 inch per second, find a formula for the time-rate at which y increases, and give the numerical result when $x = 6$.

12. A vessel containing water is in the form of an inverted hollow cone with vertical angle 90°. If the depth of water be x feet, what is the volume of water? (The volume of a cone is $\frac{1}{3}$ of that of a cylinder of the same base and height.)

If water flows in at the rate of 1 cubic foot per minute, at what rate is the level of water rising when the depth is 2 feet?

13. The volume of a spherical cap of height h (i.e. that portion of a sphere cut off by a plane distant $r - h$ from the centre) is given by the formula $\pi h^2 \left(r - \frac{h}{3}\right)$, where r is the radius of the sphere.

Use this result to solve the following problem:—

A hemispherical bowl has a diameter of 4 feet. Water is pouring into it from a tap at the rate of 33 cubic inches a second. At what rate (inches per second) is the depth of the water in the bottom of the bowl rising at the instant when this depth is 6 inches?

14. A vessel containing water is in the form of an inverted hollow pyramid; its base is a square of side 6 feet, and altitude of pyramid is 10 feet. If depth be x feet, what is the volume of water? (Volume of pyramid $= \frac{1}{3}$ base $\times$ height.)

If water flows in at the rate of 10 cubic feet per minute, at what rate is the level of water rising when the depth is 4 feet?

15. The gas-bag of an airship consists of a cylindrical middle and two hemispherical ends. As it is inflated the total length from end to end always remains three times the diameter of the cross section of the cylindrical portion.

If the radius of cross-section is r metres, and is increasing with a speed of v metres per second, find a formula giving the rate of increase of the volume in cubic metres per second, at this instant.

If air is being forced in at the uniform rate of $\cdot 1$ cubic metre per second, at what rate in centimetres per second is the radius increasing at the instant when radius $= 2$ metres?

16. If $pv = 100$, prove that the rate of change of p compared with that of v is always negative. Explain this.

17. A cube, length of side x, has volume V. Find $\frac{dV}{dx}$.

A cube of side 12 ins. is heated uniformly so that each side becomes $12\cdot 3$ ins. Use above result to find approximately the increase in volume. What is the exact value of the increase in volume?

18. A uniform sphere diameter $2\frac{1}{16}$ inches is turned down uniformly to diameter $2\frac{1}{32}$. What is approximately the percentage by which its weight is reduced?

19. The curve whose equation is

$$y = ax^3 + bx^2 + cx$$

(a, b, and c constants) has a slope of 45° at the origin where $x = 0$ and $y = 0$ and touches Ox at the point ($x = 1$, $y = 0$).

Find from these data the values of a, b, c.

20. An electric current C is changing according to the law

$$C = 20 + 21t - 14t^2.$$

where t is number of seconds. The voltage V is such that $V = RC + L\dfrac{dC}{dt}$; find V as a function of the time where R and L are constants.

CHAPTER XXVI

INTEGRATION

INTEGRATION THE INVERSE OF DIFFERENTIATION.

255. If $y = x^3$, then $\dfrac{dy}{dx} = 3x^2$. The latter statement is called a **differential equation**, i.e. an equation involving one or more derived functions.

If we are told that $\dfrac{dy}{dx} = 3x^2$, what information have we about y? Clearly we are *not* told directly that y is a certain function of x. We are told that the derived function of y is a certain function of x. Can we from this find y as a function of x?

In differentiation, we solve the problem: given a function, find the derived function.

The inverse problem—given the derived function, find the original function—is the problem of **integration.**

To solve the differential equation

$$\frac{dy}{dx} = 3x^2$$

we have to solve a problem of integration.

256. Now it is clear that *a* solution of this differential equation is

$$y = x^3.$$

But is this the only solution possible? What is the value of $\frac{d}{dx}(x^3+1)$? of $\frac{d}{dx}(x^3+2)$? Can you suggest a more general solution than x^3?

It will be seen that $y = x^3 + c$, where c is a constant, satisfies the differential equation $\frac{dy}{dx} = 3x^2$. This statement is to be verified by differentiating $x^3 + c$. As regards the constant c, any value will do; c is called an **arbitrary** constant.

257. Given the fact that $\frac{dy}{dx} = 3x^2$, we are given the gradient of a graph for every value of x; the problem of integration is the problem of deducing the graph. Why do we get a solution involving an arbitrary constant? If we compare the various curves obtained by giving different values to c, e.g. $y = x^3$, $y = x^3 + 1$, $y = x^3 + 2$, $y = x^3 - 1$, etc., we see that they are really the same curve, moved higher up or lower down on the paper. Thus $y = x^3 + 1$ differs from $y = x^3$ simply in this, that the whole curve lies 1 unit higher. Now this shifting of the whole curve up or down does not alter the gradient for any value of x; it alters y, but does not alter $\frac{dy}{dx}$. This explains why every graph obtained by giving different values to c in $y = x^3 + c$ should satisfy the same condition as regards gradient.

EXERCISE XXVI. a.

Solve the following differential equations:—

1. $\frac{dy}{dx} = 2x$.

2. $\frac{dy}{dx} = x$.

3. $\frac{dy}{dx} = ax$.

4. $\frac{dy}{dx} = x^2$.

5. $\frac{dy}{dx} = ax^2$.

6. $\frac{dy}{dx} = px^2 + qx + r$.

7. $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$.

8. $\dfrac{dy}{dt} = \dfrac{2}{t^2}$.

9. $\dfrac{dy}{dt} = \dfrac{a}{t^2} + b$.

10. $\dfrac{dx}{dt} = (t-1)(t-2)$.

11. $\dfrac{dy}{dx} = x^2\left(2 - \dfrac{3}{x^2} + \dfrac{4}{x^4}\right)$.

12. $\dfrac{dy}{dx} = \dfrac{(x+1)(x-1)}{x^2}$.

258. Given velocity-time law, to find space-time law. In the case of a point moving in a straight line, if we know the law connecting distance with time, we can find the velocity, and hence the acceleration, by a process of differentiation.

In the inverse problem, we may be given the law connecting velocity and time, and required to find the distance. If the law is the simplest possible, i.e. that of uniform velocity, we can use arithmetic to find the distance. But suppose that the velocity is known to vary as the time; i.e. $v = kt$; or in other words

$$\frac{ds}{dt} = kt.$$

The solution of this differential equation is

$$s = \tfrac{1}{2}kt^2 + c \qquad \text{(verify by differentiating).}$$

We cannot find the value of c unless we have some further information; e.g. the position of the point at some instant of time. Suppose that the point is at the 2 ft. mark when $t = 0$.

Then $$2 = 0 + c, \quad \therefore c = 2,$$
$$\therefore s = \tfrac{1}{2}kt^2 + 2.$$

EXERCISE XXVI. b.

Find the law connecting time and distance when the velocity time law is given, as follows:—

1. $v = 2 + 3t$; and $s = 3$ when $t = 0$.
2. $v = t^2 + 4t - 5$; and $s = 4$ when $t = 1$.
3. $v = 2 - \dfrac{1}{t^2}$; and $s = 3$ when $t = 1$.

4. $v = t^2 + \frac{1}{t^2}$; and $s = 4$ when $t = 2$.

5. $v = (t-1)(t-3)$; and $s = 5$ when $t = 0$.

6. $v = \frac{t^2 - 1}{t^2}$; and $s = 6$ when $t = 1$.

7. A stopping train runs between two consecutive stations with a velocity given by the law $\frac{1}{2}t(2-t)$, velocity being measured in miles per minute and time being reckoned in minutes from the first station. Show

(i) that the time taken between the two stations is 2 minutes;

(ii) that the maximum velocity attained is $\frac{1}{2}$ mile per minute, or 30 miles per hour;

(iii) that the distance between the two stations is $\frac{2}{3}$ mile.

259. Given acceleration-time law, to find velocity-time and space-time laws. Suppose that the acceleration (f units) of a point moving in a straight line is known to follow the law $a + bt$.

Then
$$f = \frac{dv}{dt} = a + bt,$$
$$\therefore\ v = at + \tfrac{1}{2}bt^2 + c.$$

To find c we must have another datum. Suppose that $v = 3$ when $t = 0$; then
$$3 = 0 + 0 + c,$$
$$\therefore\ v = at + \tfrac{1}{2}bt^2 + 3,$$
and the velocity-time law is now known.

From this we may find the space-time law.

EXERCISE XXVI. c.

1. A stone is thrown vertically downwards with a velocity of 100 ft. per sec., and the downward acceleration due to gravity is 32 ft. per sec. per sec. Find the velocity t seconds later, and also the distance covered.

2. A stone is thrown vertically upwards with a velocity of 100 ft. per sec., and the downward acceleration due to gravity is 32 ft. per sec. per sec. Find the velocity t seconds later, and also the distance covered.

3. A body moves in a straight line in such a way that its acceleration t seconds after it has passed a fixed point O on the line is $3t$ ft. per sec. per sec. Find its distance from O after 5 seconds, given that its speed when it passes O is 10 ft./sec.

4. A body moves in a straight line under an acceleration governed by the law $f = 2 + 6t$, the units of space and time being feet and seconds respectively. When $t = 0$, $s = 4$ and $v = -2$. How far does the body move during the 3rd second? Show that the velocity is a minimum when the acceleration is zero, and that this velocity is $2\frac{1}{3}$ ft. per sec. in the negative direction.

5. A body moves in a straight line, and the acceleration is given as the following function of the time: $3t^2 + t - 2$: and $v = 10$ when $t = 1$; investigate the minimum and maximum values of the velocity.

Also show that the body is instantaneously at rest at some time between $t = -2$ and $t = -3$.

Area under a Graph.

260. Take the graph $y = \frac{1}{2} + x^2$. Suppose that it is required to find the area ARSB bounded by the graph, the x-axis, and the ordinates AR and BS. The positions of these ordinates are defined by OA $= a$ units, OB $= b$ units.

Consider the shaded area ARPN; and let ON $= x$ units. As NP moves to the right, the shaded area grows; when $x = a$, the shaded area is zero, and when $x = b$ it coincides with the area ARSB. The growth of the shaded area depends on the movement of NP; i.e. on the growth of ON, which is x units. The

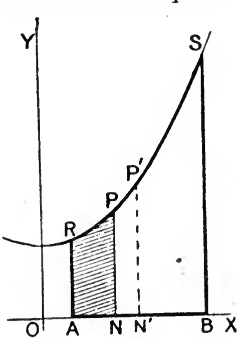

Fig. 54.

shaded area is therefore a function of x. We have to find what function.

Let the shaded area contain z units of area.

Let P be moved to P′, so that ON is increased by $NN' = \Delta x$ units, and x is increased to $x + \Delta x$. The area z is increased by the strip NPP′N′; call this Δz units of area.

Now, in fig. 54, the area of this strip

$>$ rectangle of base NN′ and height NP,

and $<$ " " " " " N′P′.

$\therefore$ Δz lies between $\Delta x \times NP$ and $\Delta x \times N'P'$,

$\therefore$ $\dfrac{\Delta z}{\Delta x}$ lies between NP and N′P′.

Now, as $\Delta x \to 0$, $\dfrac{\Delta z}{\Delta x} \to \dfrac{dz}{dx}$.

But $\dfrac{\Delta z}{\Delta x}$, lying between NP and N′P′, tends to equality with NP as $\Delta x \to 0$.

Therefore $$\frac{dz}{dx} = NP = y = \tfrac{1}{2} + x^2,$$

$$\therefore \text{ integrating, } z = \frac{x}{2} + \frac{x^3}{3} + c.$$

To find c, note that when $x = a$, NP coincides with AR, and $z = 0$.

$$\therefore 0 = \frac{a}{2} + \frac{a^3}{3} + c, \qquad \therefore c = -\frac{a}{2} - \frac{a^3}{3}.$$

Hence $$z = \frac{x}{2} + \frac{x^3}{3} - \frac{a}{2} - \frac{a^3}{3}.$$

When $x = b$, the shaded area coincides with ARSB; this area is therefore $\dfrac{b}{2} + \dfrac{b^3}{3} - \dfrac{a}{2} - \dfrac{a^3}{3}$.

Note that the result $\frac{d}{dx}(\text{area}) = y$ is true whatever function y is of x. In finding areas of graphs, we may start by assuming this result, unless we are required to proceed from first principles.

EXERCISE XXVI. d.

1. Find the area bounded by the parabola $y = 3 + 4x + 3x^2$, the axis of x, and the ordinates at $x = 1$ and $x = 2$.

2. P is a point on the parabola $y = x^2$; PN is drawn perpendicular to the x axis; O is the origin. Prove that the area bounded by PN, ON, and the curve is $\frac{1}{3}$ of that of the rectangle whose sides are ON, PN.

3. Find the area enclosed between the curve $y = \frac{10}{x^2}$, the x axis and the ordinates at $x = 2$, $x = 8$.

4. Find the area enclosed between the axis of x and the curve $y = 7x - x^2 - 10$. (At what points does the curve cut the x axis?)

5. Find the area included between the curve whose equation is

$$x^2y = x^3 + a^3,$$

the axis of x, and the ordinates at $x = a$ and $x = 2a$.

6. Prove that the area of the curve $y = a + bx + cx^2$ between the ordinates at $x = -d$ and $x = +d$ and the axis of x is $2ad + \frac{2}{3}cd^3$.

Hence show that if h_1, h_2 and h_3 are the ordinates of this curve at the points $x = -d$, $x = 0$, and $x = +d$ respectively, the area in question is equal to $\frac{d}{3}(h_1 + 4h_2 + h_3)$.

7. In the curve whose equation is

$$y = 5x - x^2,$$

at what rate with respect to x is the area between the curve, the axis Ox, and an ordinate PM increasing as M moves along the axis Ox?

If M is moving at 3 feet per second and all dimensions are in feet give the rate of increase of area per second when $x = 1$.

8. Where does the curve $y = x - \dfrac{1}{x^2}$ cut the axis of x? Find the area below the curve from this point to $x = 2$.

If x is increasing at a rate v, at what rate is the area growing?

9. Find the area bounded by the curve $y = x^2 - \dfrac{1}{x^2}$, the parabola $y = x^2$, and the ordinates at $x = a$, $x = b$.

Volume of Solid of Revolution.

261. If any line (straight or curved) be made to rotate about a straight line, it generates a surface called a **surface of revolution.** Thus, a straight line rotating about a parallel straight line generates a circular cylinder; a straight line rotating about an intersecting line generates a cone; a circle rotating about a diameter generates a sphere; a parabola rotating about its axis generates a paraboloid of revolution; an ellipse rotating about either axis generates an ellipsoid of revolution, or a spheroid. The corresponding solids are called **solids of revolution.**

As an example of the method of finding the volume of a solid of revolution, we will find the volume cut off by a plane perpendicular to the axis from the cone generated by rotating the straight line $y = \frac{1}{2}x$ about the axis of x.

The x axis is also the axis of the cone; let us find the volume cut off by a plane drawn perpendicular to the x axis at the point B given by $OB = b$ units.

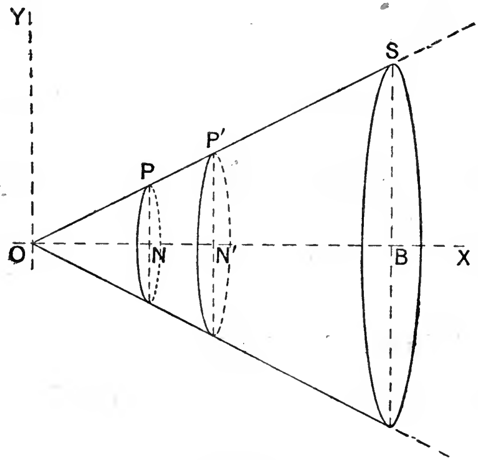

Fig. 55.

Consider the conical volume, z units of volume, cut off by a plane perpendicular to the axis drawn through N, where $ON = x$ units. As this plane moves to the right, x increases and so does z; z is a function of x, which remains to be determined.

Let ON be increased by a length $NN' = \Delta x$ units, so that x is increased to $x + \Delta x$. The volume z units is increased by the disc whose plane faces are circles with centres at N and N′: call this Δz units of volume.

Now the volume of the disc is greater than that of a cylinder whose base is the circle with centre N and whose height is Δx. Similarly the volume of the disc is less than that of a cylinder whose base is the circle with centre N′ and whose height is Δx.

$$\therefore\ \Delta z \text{ lies between } \pi PN^2 . \Delta x \text{ and } \pi P'N'^2 . \Delta x,$$

$$\therefore \frac{\Delta z}{\Delta x} \text{ lies between } \pi \text{PN}^2 \text{ and } \pi \text{P}'\text{N}'^2.$$

Now, as $\Delta x \to 0$, $\dfrac{\Delta z}{\Delta x} \to \dfrac{dz}{dx}$.

But $\dfrac{\Delta z}{\Delta x}$, lying between πPN^2 and $\pi\text{P}'\text{N}'^2$, tends to equality with πPN^2 as $\Delta x \to 0$.

$$\therefore \frac{dz}{dx} = \pi\text{PN}^2 = \pi y^2 = \frac{\pi x^2}{4},$$

$$\therefore z = \frac{\pi x^3}{12} + c.$$

To find c, note that when $x=0$, $z=0$, $\therefore c=0$,

$$\therefore z = \frac{\pi x^3}{12}.$$

When $x=b$, we have the volume of the cone the centre of whose base is B; viz. $\dfrac{\pi b^3}{12}$.

This is $\dfrac{1}{3}\cdot\pi\left(\dfrac{b}{2}\right)^2\cdot b = \dfrac{1}{3}\pi\text{BS}^2\cdot\text{OB}$; in other words, $\dfrac{1}{3}$ of the cylinder of the same base and height.

The result $\dfrac{d}{dx}$ (volume) $= \pi y^2$ applies to the volume generated by the rotation of a curve about the axis of x, whatever function y is of x.

EXERCISE XXVI. e.

1. $y^2 = 4ax$ represents a parabola whose axis of symmetry is the x-axis. By rotating the curve about the x-axis, a paraboloid of revolution is formed. Of what form are sections of this surface made by planes perpendicular to the x-axis? Of what form are sections made by planes containing the x-axis? Find the volume bounded by the paraboloid and a plane perpendicular to the x-axis at $x=b$. Prove that this is $\frac{1}{2}$ the volume of a circumscribing cylinder.

2. $y^2 = c^2 - x^2$ represents a circle of radius c, with centre at the origin. What surface is formed by rotating this about the x-axis? Find the volume bounded by this surface, and planes perpendicular to the x-axis through $x = a$, $x = b$, where a and b are positive and $b > a$. What values must be given to a and b in order to deduce the volume of the hemisphere? What is this volume?

3. Fig. 56 shows a part of the graph $y = \dfrac{1}{x}$. Find the volume generated by the rotation of APQB about the x-axis, where OA $= a$, OB $= b$. Show that as $b \to \infty$, this volume tends to a finite limit.

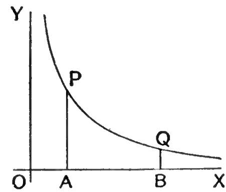

Fig. 56.

4. $y^2 = 1 - \dfrac{x^2}{4}$ represents an ellipse with centre at the origin, and longer axis along the x-axis. Find where the ellipse cuts the two axes, and hence make a rough sketch of the curve. Find the volume of the ellipsoid of revolution (prolate spheroid) formed by rotating this ellipse about its longer axis.

VOLUME OF A PYRAMID.

262. The method of § 261 may be applied to finding the volume of any solid in which the area of the plane cross-section perpendicular to a certain straight line is a known function of the distance along that line. Thus, we will find the volume of a pyramid, on a base of any shape.

Let the area of the base be S units of area, and the height h units of length.

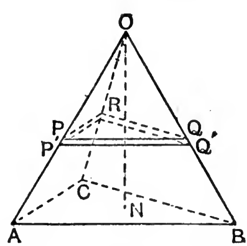

Fig. 57.

Consider a plane section of the pyramid, parallel to the base, giving the figure PQR. This is a figure similar to the base.

Let the distance of PQR from O be x units. The volume of the pyramid cut off by PQR is a function of x; let it be z units of volume.

Let PQR be moved parallel to itself to P′Q′R′, so that x is increased to $x+\Delta x$. The volume z is increased by the slice whose upper and lower faces are PQR, P′Q′R′: call this Δz units of volume.

Now the volume of this slice is greater than that of a prism whose base is PQR and whose height is Δx. Similarly the volume of the slice is less than that of a prism whose base is P′Q′R′ and whose height is Δx.

$\therefore$ Δz lies between $\Delta x \times$ area PQR and $\Delta x \times$ area P′Q′R′,

$\therefore$ $\dfrac{\Delta z}{\Delta x}$ lies between area PQR and area P′Q′R′.

Now, as $\Delta x \to 0$, $\dfrac{\Delta z}{\Delta x} \to \dfrac{dz}{dx}$.

But $\dfrac{\Delta z}{\Delta x}$, lying between areas PQR and P′Q′R′, tends to equality with PQR as $\Delta x \to 0$.

$$\therefore \frac{dz}{dx} = \text{area of PQR}.$$

But PQR is similar to ABC; and the ratio of the linear dimensions is $x : h$.

$$\therefore \text{area of PQR} = \frac{x^2}{h^2} \times \text{area of ABC}$$

$$= \frac{x^2}{h^2} \times \mathrm{S}.$$

$$\therefore \frac{dz}{dx} = \frac{x^2}{h^2} \times \mathrm{S} = \frac{\mathrm{S}}{h^2} \times x^2.$$

Observing that $\dfrac{\mathrm{S}}{h^2}$ is a constant factor, we have

$$z = \frac{1}{3}\frac{\mathrm{S}}{h^2} \times x^3 + c.$$

To find c, note that when $x=0$, $z=0$, $\therefore\ c=0$,

$$\therefore\ z=\frac{1}{3}\frac{\mathrm{S}x^3}{h^2}.$$

This gives the volume cut off from the pyramid by PQR. For the whole volume of the pyramid we make PQR coincide with ABC, and $x=h$.

$$\therefore \text{ volume of pyramid} = \tfrac{1}{3}\mathrm{S}h \text{ units of volume.}$$

Hence the important result that the volume of a pyramid is measured by one-third of the product of the base and height.

EXERCISE XXVI. f.

1. A vessel for holding water is 10 ins. deep, and the area, y sq. ins., of the water surface when the depth of the water is x ins. is given by the equation $y=2+x+\frac{1}{2}x^2$.
How much water will the vessel hold?

2. A bath is of such dimensions that, when it is filled to a depth x ft., the area of the water surface, in sq. ft., is

$$10+4x+\frac{x^2}{3}.$$

Find the number of gallons of water in the bath when it is filled to a depth of 2 ft., taking 1 cu. ft. $=6\frac{1}{4}$ gallons.

3. The shape of a hill is such that, at a height of x ft. above the foot of the hill, the horizontal section has an area of $20000\,(100-x)$ sq. ft. Find the height of the hill by calculating the height at which the sectional area is zero; also find the number of cubic feet of earth in the hill.

4. The length of a submarine boat is $2a$ ft., and the vertical height of the hull is $2b$ ft. The area of a horizontal section, at a height of x ft. above the lowest point of the hull, is

$$\pi ab\left[1-\frac{(x-b)^2}{b^2}\right] \text{ sq. ft.}$$

Prove that the volume of the hull is $\frac{4}{3}\pi ab^2$ cu. ft.

5. The height of a beer-barrel, standing on its end, is $2h$ ft., the radius of each end r ft., the radius of the greatest horizontal section R ft. (all inside measurements); when the depth of beer in the barrel is x ft., the area of the surface of the beer, in sq. ft., is

$$\pi\left\{\mathrm{R}^2-\frac{(\mathrm{R}^2-r^2)(x-h)^2}{h^2}\right\}.$$

Prove that the capacity of the barrel, in cu. ft., is

$$\frac{2\pi h}{3}(2\mathrm{R}^2+r^2).$$

MISCELLANEOUS EXERCISES

1. Find the average gradient of x^3 (i) between 1 and 3, (ii) between $1\frac{1}{2}$ and $2\frac{1}{2}$.

2. P is the point (2, 2) on the curve $y = x^3 - 3x$. Q is the point where $x = 2 + h$, and R the point where $x = 2 - h$. Find the gradient of the line QR. Explain how the result enables you to obtain the gradient of the tangent to the curve at P, putting down your reasoning in full.

3. From first principles, find the gradient of $3 - 5x$.

4. From first principles, find the gradient of $a - kx$.

5. From first principles, find the gradient of $3x^3$ at the point where $x = 2$.

6. From first principles, find the gradient of $y = 2x - x^2$ at each of the points where the curve cuts the axis of x.

7. From first principles, find the gradient of $2x + \frac{1}{x}$ at the point where $x = a$.

8. Find the function giving the gradient of

$$\text{(i)} \quad (x-1)(x-2)(x-3), \qquad \text{(ii)} \quad x\left(x - \frac{1}{x}\right)^2.$$

9. Evaluate

$$\text{(i)} \quad \frac{d}{dt}(4t^3 - 3t^2 + 5t - 6), \qquad \text{(ii)} \quad \frac{d}{dx}[(x+1)^2 + (x-1)^2].$$

10. Find the gradients of $(x+1)(3-x)$ and $\frac{4}{x}$ at the point at which $x=2$.

11. Write down the derived function of

$$\text{(i)}\quad 2x^2-\frac{3}{x}, \qquad \text{(ii)}\quad (x-1)(x-3);$$

and find the gradient of each graph at the point where $x=-1$.

12. Prove that the graphs $y=2x^2-x$ and $y=\frac{3}{x}+14$ have a common point where $x=3$. Find the gradient of each graph at this point.

13. Differentiate

$$\text{(i)}\quad x^2\left(x+\frac{3}{x^3}\right), \qquad \text{(ii)}\quad (2-3x)^2.$$

Find the points where the gradient of each graph is zero.

14. At what points does the tangent to the curve

$$y=x^3-x^2$$

make an angle of 45° with the x axis?

15. If P is a point on the curve $y=x^3$ and the tangent at P meets Ox in Q and Oy in R, express OQ in terms of the x coordinate of P and show that PR = 3PQ.

16. What is the gradient at the point (x', y') of the curve $6y=\frac{4-3x}{x}$?

The tangent at the point on this curve where $x=1$ meets the axes at the points A and B. Find the area of the triangle AOB, where O is the origin.

17. If $y=\frac{100}{x}$ what is $\frac{dy}{dx}$? Show that if the tangent at P to this curve meets the axes in A and B respectively, the triangle AOB contains 200 units of area.

Check the result by taking a simple special case.

18. Find the gradient of the curve whose equation is

$$y = x^3 - 3x^2 + 5$$

at the point where $x = 3$.

If the tangent at this point meets the axis Ox in T, find the distance OT, all dimensions being in inches.

19. Show by means of the value of $\frac{dy}{dx}$ that in the curve

$$y = 2x^3 - 3x^2 + 4$$

as x increases from 0 onwards the value of y at first decreases, but subsequently begins to increase and continues to do so. Show also that the slope of the curve may have any positive value, but that certain negative values, for instance -2, are not possible.

20. A point moves in a straight line according to the law $s = 2t^2 - 5t + 7$. Find the velocity when $t = -1$.

21. The distance of a point moving in a straight line from a fixed origin on the line is $2(t-3)(t+2)$ feet when t seconds have elapsed after a certain instant. Where is the point at that instant (i.e. when $t = 0$)? Find the velocity and acceleration after 5 seconds.

22. In a straight line motion the distance s feet from the origin of the moving body at t seconds from the start is given by $s = 8 - 75t + t^3$.

Find the velocity after 3 seconds and the average velocity during the fourth second. Find also the acceleration after 3 seconds.

What happens when $t = 5$?

23. Investigate the maxima or minima of the function

$$2 - 3x + \tfrac{1}{2}x^2.$$

24. Investigate the maxima and minima of

$$2x^3 + 3x^2 - 36x + 10.$$

25. Examine the turning values of the function

$$5 + 9x - 3x^2 - x^3.$$

26. Find what values of x make $\dfrac{1+x^3}{2} - \left(\dfrac{1+x}{2}\right)^3$ a maximum or minimum.

Distinguish whether the values correspond to a maximum or to a minimum, and illustrate by a sketch.

27. The sum of the two smaller sides of a right-angled triangle is 8 ins. Find the maximum area.

28. The sum of the two sides of a right-angled triangle is 6 ins.: find the turning value of the (hypotenuse)^2, and hence of the hypotenuse. Is this a maximum or a minimum?

29. The area of the surface of an open cylindrical tank whose volume is 220 cubic ft. is known to be $22\left(\dfrac{r^2}{7} + \dfrac{20}{r}\right)$ sq. ft. where r is the radius in feet. Find the smallest surface area the cylinder can have.

30. A point R is taken on the side AB of a triangle ABC of area z so that AR $= x$. AB, where $x > 1/2$. RQ is drawn parallel to BC to meet AC at Q, RH parallel to AC to meet BC at H, and QK parallel to AB to meet BC at K. Prove that the areas of ARQ and BRH are $x^2 z$ and $(1-x)^2 z$ respectively (notice that they are similar to ABC), and find in similar form the area of CQK; use these results to find the area of QRHK. Verify your result for QRHK by giving x the values 1 and 1/2.

Find the value of x which makes the trapezium QRHK greatest.

31. A cylinder of volume 100 cubic inches is to be made with a thin iron curved surface and a wooden base, and no lid. If the weight of 1 square inch of the sheet iron is three times the weight of 1 square inch of the wooden base, get an expression for the weight of the cylinder in terms of the radius. Find the radius when this weight is a minimum.

32. A solid is in the shape of a cylinder with hemispherical ends. If the volume is 500 cubic inches, find the value of r for which the total surface area is a minimum. What is this minimum area?

33. When there is no import duty on a certain commodity, the value of the amount imported in a year is £10,000,000. When an import duty of x per cent. is put on, the amount falls by $4x$ per cent. Calculate the annual yield of the duty when the tax is x per cent. Find the maximum duty which it is profitable to the State to impose.

34. The gradient of a certain curve which passes through the origin is given by the function $5-2x+x^2$; what is the equation of the curve?

35. Solve the following differential equations:—

(i) $\dfrac{dy}{dx}=2+3x+2x^2.$

(ii) $\dfrac{dy}{dx}=(2-x)(3+x)$

(iii) $\dfrac{dy}{dx}=\dfrac{3}{x^2}+\dfrac{x^2}{3}.$

(iv) $\dfrac{dy}{dx}=\dfrac{a}{x^2}+b+cx.$

(v) $\dfrac{dy}{dt}=\dfrac{t^2-2}{t^2}.$

(vi) $\dfrac{dy}{dt}=(t-1)^2+(t+1)^2.$

(vii) $\dfrac{ds}{dt}=3\dfrac{(t^2+1)^2}{t^2}.$

(viii) $\dfrac{ds}{dt}=\dfrac{4t^4-5t^3+3t^2+9}{t^2}.$

36. Find the law connecting time and distance when the velocity-time law is $v=3-5t^2$ and $s=2$ when $t=0$.

If the units of space and time are feet and seconds respectively, find how far the point travels during the 5th second. What is the average velocity during that second? What is the velocity at the middle instant of that second?

Investigate the maximum or minimum values of the velocity.

37. Find the law connecting time and distance when the velocity-time law is $v=\frac{4}{t^2}+1$ and $s=-3$ when $t=1$. When is the moving point at the origin?

38. The acceleration of a point moving along a straight line is given by the following function of the time: $5+12t$. The velocity is zero when $t=2$; and the point is at the origin when $t=-1$. Find the distance from the origin in terms of t. At what other instant is the velocity zero?

Investigate when the velocity is a maximum or minimum, and shew that this occurs at an instant midway between the instants of zero velocity.

39. Find the area enclosed by the curve $y=\frac{3}{x^2}$, the axis of x, and the ordinates at $x=1$, $x=5$.

40. The equation of a curve is $y=x+\frac{x^2}{2}$.

Find the area enclosed between the curve, the axis of x, and the ordinate at the point P at which $x=1$.

Deduce the area between the arc OP and the chord OP.

41. Find the area included between the curve $y=x^2$, the axis Ox, and the ordinate $x=10$. Where would you draw a line perpendicular to Ox to bisect the area?

42. Find the area of the segment of the parabola $y=2x^2$ cut off by the straight line $y=4x$.

43. A plane area is bounded by the straight lines

$$x=0 \text{ and } x=15,$$

and the curves
$$y=4+\frac{x}{20}-\frac{x^2}{200},$$

and
$$y=\frac{x^2}{200}-4-\frac{x}{20}.$$

Find the total area enclosed, lengths being in feet.

44. The shape of the head of a certain bullet is that formed by the revolution of part of a parabola about the axis of the parabola. If the total length of the head, measured from the vertex of the parabola, is 2·7 centimetres, and the diameter of the bullet is 0·6 centimetre, find the volume of the head.

[The equation of the parabola is $y^2 = 4ax$, with the tangent at the vertex as axis of y, and the vertex as origin.]

ANSWERS

Note that, when 4-figure logarithm tables are used, the fourth figure is generally untrustworthy.

EXERCISE XXIII. a.

PAGE

360. 1. 2 in each case. 2. $\frac{1}{2}$ in each case.
3. $-\frac{1}{3}$ in each case. 4. Gradient 5 in each case.

361. Ex. 1. 0. Ex. 2. It approaches the y axis.

362. Ex. 4. $-7, -5, -3, -1, 1, 3, 5, 7$.

363. Ex. 5. ·04, ·06, $-$·05 (very roughly).

368. Ex. 6. 4, 6, 8.

EXERCISE XXIII. b.

370. 1. $4x$. Twice the gradient of $y=x^2$.
2. $10x$. Five times that of x^2. 3. x. Half that of x^2.
4. $2ax$. 5. $2x$, $2x$, $2ax$.

371. Ex. 7. 1, 2, $\frac{1}{2}$, 3, 3, -3, b. Ex. 8. 14, 23, 7, 167.

EXERCISE XXIII. c.

374. 1. $2x$. 2. $2x+3$. 3. $6x+1$. 4. $\frac{1}{2}-\frac{3}{2}x$.
5. $-\frac{8}{3}x+4$. 6. $2ax+b$. 7. See § 257.

EXERCISE XXIII. d.

377. 1. $x=1$; 5, max. 2. $(2, -9)$. 3. Max.
4. (i) Max. at $x=1·5$, (ii) min. at $x=2$, (iii) min. at $x=-2$.

378. 5. (i) 5 and 5, (ii) 5 and 5. 6. A square.
7. $(6x-\frac{1}{2}x^2)$ sq. in.; when $x=6$.
8. $t=\frac{u}{64}$, height$=\frac{u^2}{256}$ feet; $t=1$, height$=16$ ft.
9. After 3 hrs.; 5 miles. 10. 20.

379. 11. $x=2$, min. 12. $v=30x-11x^2$; 15/11.

EXERCISE XXIII. e.

PAGE

380. 1. 2·1 cu. in.; 1 %. 2. ·000016 = $1·6 \times 10^{-5}$ ft.
4. 150 ft.; 5 ft.

382. Ex. 1. (i) 80 ft. per sec., (ii) 64 ft. per sec.,
(iii) 48 ft. per sec., (iv) 40 ft. per sec.

Ex. 2. (i) 230 yds. per min., (ii) 750 yds. per min.,
(iii) 490 yds. per min.

EXERCISE XXIV. a.

384. 1. 1·4 ft. per day.

385. 2.

Time in sec.	0	10	20	30	35	40	50
Veloc. in ft./sec.	0	6·5	13	27	37	27	0

3. 2·2, 1·8, 1·2 degrees per min.
4. 1·3, 0·5 gallons per sec.

386. 7. $v = -3 + 2t$; when $t = 1·5$; when $t > 1·5$; when $t < 1·5$; when $t = 1·5$; minimum.
8. When $t = 1$; $t > 1$; $t < 1$; $t = 1$.

EXERCISE XXIV. b.

387. 1. $2·4\pi$ (= 7·54) sq. in. per sec.
2. Where $x = 2$; where $x = 1·5$.

388. 3. $48\pi a$ sq. in. per sec.; $72\pi a$ sq. in. per sec.
4. 2·3, 1·4 in. per min. 5. $\frac{b}{2\pi a x}$ in. per sec.

EXERCISE XXIV. c.

389. 2. $t = 3\frac{3}{4}$; 300 ft.

EXERCISE XXIV. d.

390. 1. (i) 80 ft. per sec., (ii) $(60 - 32t)$ ft. per sec.; 60 ft. per sec.,
(iii) when $t = 1\frac{7}{8}$; $56\frac{1}{4}$ ft., (iv) -60 ft. per sec.
3. $(50 - \frac{4}{125}x)$ ft. per sec.; 1562·5 ft.

EXERCISE XXIV. e.

PAGE

392. 1. 0·74 ft. per sec. per sec.

2. (i) -26 ft. per sec. per sec., (ii) -22 ft. per sec. per sec.

393. 3. (i) $\frac{3}{2}$ ft., (ii) $(-2+t)$ ft. per sec., (iv) -2 ft. per sec., (v) after 2 secs., (vi) after 1 and 3 secs., (vii) 1 ft. per sec. per sec. in each case.

EXERCISE XXV. a.

396. 2. 12. 3. (i) $\frac{3}{4}$, (ii) 27. 4. $x=\pm\sqrt{\frac{1}{3}}=\pm 0{\cdot}577$.
6. $x=\pm 1$. 7. -1, 3.

EXERCISE XXV. b.

398. 1. $x^2-4x+\dfrac{3}{x^2}$. 2. (i) $-2x^2-\dfrac{1}{2x^2}-x$,

(ii) $3+12x^2-12x+\dfrac{3}{x^2}$, (iii) $\dfrac{3}{5x^2}+\frac{1}{2}x^2$. 3. (i) $6+6x^2$,

(ii) $-\dfrac{3}{x^2}+5+12x$, (iii) $12x^2+\dfrac{3}{x^2}-4x$, (iv) $4+24x+27x^2$.

4. $-\frac{1}{4}$.

399. 5. $2a-6b\theta^2$.

6. (i) $x>5$ or <1 positive; $5>x>1$ negative,
(ii) $x>3$ or $2>x>1$ positive; $3>x>2$ or $x<1$ negative,
(iii) $4>x>2$ positive; $x>4$ or $x<2$ negative,
(iv) $5>x>-3$ positive; $x>5$ or $x<-3$ negative,
(v) $x<3$ positive; $x>3$ negative,
(vi) $x>2$ or $1>x>-1$ or $x<-2$ positive;
$2>x>1$ or $-1>x>-2$ negative.

8. (i) Min. $(1, -2)$, max. $(-1, 2)$, (ii) min. $(3, 0)$, max. $(1, \frac{4}{3})$,
(iii) min. $(\frac{1}{2}, 4)$, max. $(-\frac{1}{2}, -4)$,
(iv) min. $(2, -9\frac{1}{3})$, max. $(1, -8\frac{2}{3})$, min. $(-1, 8\frac{2}{3})$, max. $(-2, 9\frac{1}{3})$.

9. $E=\frac{3}{4}\sqrt{2}=1{\cdot}06$, $x=\frac{1}{2}\sqrt{2}=0{\cdot}707$, min.

10. When $x=1$; when $x=-1$.

EXERCISE XXV. c.

PAGE

399. 1. $(14-2x)$ ft.; $x^2(14-2x)$ cu. ft.; 4·7; 102 cu. ft.

400. 2. 1·57. 3. 1·83. 4. Side of base 20 ft.; height 10 ft.

5. 379. 6. $\left(2a^2+3cx+\dfrac{4a^2c}{x}\right)$ sq. in.; 4 : 3.

401. 7. Side of base $\frac{1}{2}\sqrt[3]{4a}$ ft., height $\sqrt[3]{4a}$ ft.

8. Breadth $\frac{1}{2}$ ft.; depth 1 ft. 9. $\sqrt{2}:1$.

10. $\dfrac{1000}{9\pi}\sqrt{3\pi}=108{\cdot}6$ cu. in. 11. 2 of 2 ft., 4 of $\frac{1}{2}$ ft.

12. When it is a square.

402. 13. When height $=2\sqrt{\frac{1}{3}}\times$ radius of sphere.

15. £$\left(\dfrac{1750}{v}+2v^2\right)c$; 7·6.

16. £$\left(\frac{1}{72}x^2+\dfrac{180}{x}\right)$; 18·6 miles per hr.; £14. 10*s.*

17. $a=\frac{1}{2}$, $b=1800$; 60 miles per hr.; £60.

18. (i) feet, 2 cu. ft., (ii) 2 feet, $\dfrac{8}{\pi}=2{\cdot}55$ cu. ft.

403. 19. $\sqrt{\dfrac{b}{a}}$. 20. 9·2 knots.

EXERCISE XXV. d.

1. (i) $(96+24h+2h^2)$ ft./sec., (ii) $(96-24h+2h^2)$ ft./sec., (iii) $(96+2h^2)$ ft./sec.

404. 2. (i) 229 ft., (ii) 149 ft. per sec., (iii) 72 ft. per sec. per sec.

3. (i) -2 ft./sec., (ii) when $t=\pm 0{\cdot}913$ sec.

4. (i) 26 ft./sec., (ii) 26·5 ft./sec., (iii) 25·75 ft./sec.

5. 28 ft./sec., 26 ft. per sec. per sec.

6. (i) 63 ft./sec., (ii) 62·75 ft./sec., (iii) 27 ft. per sec. per sec.

7. (i) 41 ft., (ii) 5 ft./sec., (iii) -10 ft. per sec. per sec.

405. 8. (i) $(b+6c+28d)$ ft./sec., (ii) $(b+6c+27d)$ ft./sec., (iii) $(2c+18d)$ ft. per sec. per sec.

9. $8\pi=25{\cdot}1$ cu. in./min.

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405. 10. (i) $\frac{1}{36\pi}=\cdot 00885$ in./sec., (ii) $\frac{1}{400\pi}=\cdot 000796$ in./sec.

11. $y=\pi(24x-x^2)$; $\pi(24-2x)$ sq. in./sec.; $12\pi=37\cdot 7$ sq. in./sec.

12. $\frac{1}{3}\pi x^3$ cu. ft.; $\frac{1}{4\pi}=\cdot 0796$ ft./sec. 13. $\frac{1}{24}$ in./sec.

406. 14. $\frac{12}{100}x^3$ cu. ft.; $\frac{1000}{576}=1\cdot 74$ ft./min.

15. $16\pi r^2 v$; $\frac{10}{64\pi}=\cdot 0497$ cm./sec.

17. $3x^2$; 130 cu. in.; $132\cdot 867$ cu. in. 18. $4\frac{1}{2}$ %.

407. 19. $a=1,\ b=-2,\ c=1$.

20. $V=20R+21L+(21R-28L)t-14Rt^2$.

EXERCISE XXVI. a.

409. 1. $y=x^2+c$. 2. $y=\frac{1}{2}x^2+c$. 3. $y=\frac{1}{2}ax^2+c$.

4. $y=\frac{1}{3}x^3+c$. 5. $y=\frac{1}{3}ax^3+c$. 6. $y=\frac{1}{3}px^3+\frac{1}{2}qx^2+rx+c$.

410. 7. $y=\frac{1}{x}+c$. 8. $y=-\frac{2}{t}+c$. 9. $y=-\frac{a}{t}+bt+c$.

10. $x=\frac{1}{3}t^3-\frac{3}{2}t^2+2t+c$. 11. $y=\frac{2}{3}x^3-3x-\frac{4}{x}+c$.

12. $y=x+\frac{1}{x}+c$.

EXERCISE XXVI. b.

1. $s=3+2t+\frac{3}{2}t^2$. 2. $s=\frac{1}{3}t^3+2t^2-5t+\frac{20}{3}$. 3. $s=2t+\frac{1}{t}$.

411. 4. $s=\frac{1}{3}t^3-\frac{1}{t}+\frac{11}{6}$. 5. $s=\frac{1}{3}t^3-2t^2+3t+5$.

6. $s=t+\frac{1}{t}+4$.

EXERCISE XXVI. c.

1. $(100+32t)$ ft./sec.; $(100t+16t^2)$ ft.

412. 2. $(100-32t)$ ft./sec.; $(100t-16t^2)$ ft. 3. $112\frac{1}{2}$ ft.

4. 22 ft.

5. When $t=\frac{2}{3}$, min. $=9\cdot 685$; when $t=-1$, max. $=12$.

EXERCISE XXVI. d.

PAGE

414. 1. 16. 3. $3\frac{3}{4}$. 4. $4\frac{1}{2}$. 5. $2a^2$.

415. 7. $\dfrac{dA}{dx}=5x-x^2$; 12 sq. ft./sec. 8. At $x=1$; 1; $\left(x-\dfrac{1}{x^2}\right)v$.

9. $\dfrac{1}{a}-\dfrac{1}{b}$.

EXERCISE XXVI. e.

417. 1. $2\pi ab^2$.

418. 2. $\pi c^2(b-a)-\pi\dfrac{b^3-a^3}{3}$; $a=0$, $b=c$; $\frac{2}{3}\pi c^3$.

3. $\pi\left(\dfrac{1}{a}-\dfrac{1}{b}\right)$; $\pi\dfrac{1}{a}$. 4. $\frac{8}{3}\pi$.

EXERCISE XXVI. f.

420. 1. $236\frac{2}{3}$ cu. in. 2. $180\frac{5}{9}$ gallons.
3. 100 ft.; 100 million cu. ft.

MISCELLANEOUS EXERCISES.

422. 1. (i) 13, (ii) 12·25. 2. $9+h^2$; 9. 3. -5.
4. $-k$. 5. 36. 6. At $x=0$, 2; at $x=2$, -2.

7. $2-\dfrac{1}{a^2}$. 8. (i) $3x^2-12x+11$, (ii) $3x^2-2-\dfrac{1}{x^2}$.

9. (i) $12t^2-6t+5$, (ii) $4x$.

423. 10. -2, -1. 11. (i) $4x+\dfrac{3}{x^2}$, -1, (ii) $2x-4$, -6.

12. 11, $-\frac{1}{3}$.

13. (i) $3x^2-\dfrac{3}{x^2}$, when $x=\pm1$, (ii) $18x-12$, when $x=\frac{2}{3}$.

14. At the points where $x=1$ and $x=-\frac{1}{3}$. 15 $OQ=\frac{2}{3}x$.

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423. 16. $-\frac{2}{3x'^2}$, $\frac{25}{48}$. 17. $-\frac{100}{x^2}$.

424. 18. 9, $2\frac{4}{9}$. 20. -9.

21. 12 ft. from the fixed origin in the negative direction; 18 ft./sec. ; 4 ft./sec^2.

22. -48 ft./sec., -38 ft./sec., 18 ft./sec.^{2}; when $t=5$, the velocity is zero.

23. At $x=3$, min. $=-\frac{5}{2}$.

24. At $x=2$, min. $=-34$; at $x=-3$, max. $=91$.

425. 25. At $x=1$, max. $=10$; at $x=-3$, min. $=-22$.

26. At $x=1$, min. $=0$; at $x=-\frac{1}{3}$, max. $=\frac{4}{9}$. 27. 8 sq. in.

28. $3\sqrt{2}=4{\cdot}24$ in. ; min. 29. 160 sq. ft.

30. $(1-x)^2z$, $(4x-1-3x^2)z$; $x=\frac{2}{3}$.

31. $\left(\pi r^2+\frac{600}{r}\right)k$, where k is the weight of 1 sq. in. of the wood; $r=\sqrt[3]{\frac{300}{\pi}}=4{\cdot}57$ in.

426. 32. $r=\sqrt[3]{\frac{3000}{8\pi}}=4{\cdot}92$ in. ; 305 sq. in.

33. £$(100{,}000x-4000x^2)$; $12\frac{1}{2}$ per cent.

34. $y=5x-x^2+\frac{1}{3}x^3$.

35. (i) $y=2x+\frac{3}{2}x^2+\frac{2}{3}x^3+c$. (ii) $y=6x-\frac{1}{2}x^2-\frac{1}{3}x^3+c$.

(iii) $y=-\frac{3}{x}+\frac{x^3}{9}+c$. (iv) $y=-\frac{a}{x}+bx+\frac{cx^2}{2}+k$.

(v) $y=t+\frac{2}{t}+c$. (vi) $y=\frac{2}{3}t^3+2t+c$.

(vii) $s=t^3+6t-\frac{3}{t}+c$. (viii) $s=\frac{4}{3}t^3-\frac{5}{2}t^2+3t-\frac{9}{t}+c$.

36. $s=2+3t-\frac{5}{3}t^3$; $-98\frac{2}{3}$ ft.; $-98\frac{2}{3}$ ft./sec.; $-98\frac{1}{4}$ ft./sec.; max. when $t=0$.

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427. 37. $s=-\frac{4}{t}+t$; when $t=\pm 2$.

38. $s=-\frac{69}{2}-34t+\frac{5}{2}t^2+2t^3$; when $t=2$ or $-\frac{17}{6}$; min. when $t=-\frac{5}{12}$.

39. $2\frac{2}{5}$ units of area.   40. $\frac{2}{3}$, $\frac{1}{12}$ units of area.

41. $333\frac{1}{3}$ sq. cm.; at $x=\sqrt[3]{500}=7{\cdot}94$.

42. $\frac{8}{3}$ units of area.   43. 120 units of area.

428. 44. $0{\cdot}1215\pi={\cdot}3817$ cu. cm.

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